Math 312, Autumn 2007
Problem Set 6

Rudin, Chapter 6: 1,3,4,9,10 (this is very long, the equivalent of about three problems), 11, 12, 13

Exercise 1 *Suppose that* $X_1, X_2, \ldots$ *are independent, mean zero, variance one random variables not necessarily identically distributed. Let* $Z_n = (X_1 + \cdots + X_n)/\sqrt{n}$. *Give an example to show that it is not necessarily true that for all* $a < b$,

$$\lim_{n\to\infty} \mathbf{P}\{a \le Z_n \le b\} = \int_a^b \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}\, dx. \tag{1}$$

(Hint: try a modification of the example in Exercise 4.5 of the probability notes.)

Exercise 2 *Suppose in the last exercise we also assume that there is a* $K < \infty$ *such that* $\mathbf{E}[|X_j|^3] \le K$ *for all* j. *Let* ϕ_j *denote the characteristic function of* X_j.

1. *Show that there is a* c *such that for all* j *and all* $t \in \mathbb{R}$,

$$\left|\phi_j(t) - \left[1 - \frac{t^2}{2}\right]\right| \le c\,|t|^3.$$

2. *Show that the central limit theorem holds in this case, i.e., for all* $a < b$, *(1) holds.*