Math 312, Autumn 2007
Problem Set 7

Rudin, Chapter 7: 1,2,5,6,7,12,13,19

Exercise 1 *Suppose $X_1, X_2, X_3, \ldots$ are independent random variables with mean zero and variance σ^2. Let $S_n = X_1 + \cdots + X_n$ and let $\mathcal{F}_n$ be the σ-algebra generated by $X_1, \ldots, X_n$.*

1. *If $m < n$, find $\mathcal{E}[S_n^2 \mid \mathcal{F}_m]$.*
2. *Suppose $\mathbf{E}[X_j^3] = 0$ for all j. If $m < n$, find $\mathcal{E}[S_n^3 \mid \mathcal{F}_m]$.*
3. *Suppose for each m, B_m is a bounded $\mathcal{F}_{m-1}$-measurable random variable. (We assume that B_1 is a constant random variable.) Let*
$$W_n = \sum_{j=1}^{n} B_j X_j.$$
If $m < n$, find $\mathcal{E}[W_n \mid \mathcal{F}_m]$ and $\mathcal{E}[W_{m+1}^2 \mid \mathcal{F}_m]$.

Exercise 2 *Suppose $(X, \mathcal{F}, \mathbf{P})$ is a probability space and $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3 \subset$ is an increasing sequence of sub-σ-algebras of $\mathcal{F}$. A random variable $T : \Omega \to \{0, 1, 2, \ldots\} \cup \{\infty\}$ is called a stopping time if for each n, the event $\{T \leq n\}$ is in $\mathcal{F}_n$. Let $\mathcal{F}_T$ denote the set of events $A \in \mathcal{F}$ such that for each n, $A \cap \{T \leq n\} \in \mathcal{F}_n$. Show that $\mathcal{F}_T$ is a σ-algebra.*

Exercise 3 *Suppose μ is a finite, positive Borel measure on $\mathbb{R}^k$. We will call μ d-*dimensional *if*
$$\lim_{r \to 0} \frac{\log \mu(\mathcal{B}(x, r))}{\log r} = d, \quad a.e.(\mu).$$

1. *Let μ denote the Cantor measure (the measure generated by the Cantor function). Show that μ is d-dimensional for some $0 < d < 1$ and find d.*
2. *Show that if μ is absolutely continuous with respect to Lebesgue measure, then μ is a k-dimesional measure.*
3. *Show that if $\mu(\mathbb{R}^k \setminus V) = 0$ for a countable set V, then μ is 0-dimensional.*
4. *(Harder) Construct examples to show that the converses of the last two statements are false. (You may restrict to $k = 1$.)*