Exercise 1 Suppose $X_1, X_2, X_3, \ldots$ are independent random variables with mean zero and variance $\sigma^2$. Let $S_n = X_1 + \cdots + X_n$ and let $\mathcal{F}_n$ be the $\sigma$-algebra generated by $X_1, \ldots, X_n$.

1. If $m < n$, find $\mathbb{E}[S_n^2 | \mathcal{F}_m]$.

2. Suppose $\mathbb{E}[X_j^3] = 0$ for all $j$. If $m < n$, find $\mathbb{E}[S_n^3 | \mathcal{F}_m]$.

3. Suppose for each $m$, $B_m$ is a bounded $\mathcal{F}_{m-1}$-measurable random variable. (We assume that $B_1$ is a constant random variable.) Let

$$W_n = \sum_{j=1}^{n} B_j X_j.$$  

If $m < n$, find $\mathbb{E}[W_n | \mathcal{F}_m]$ and $\mathbb{E}[W_{m+1}^2 | \mathcal{F}_m]$.

Exercise 2 Suppose $(X, \mathcal{F}, \mathbb{P})$ is a probability space and $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3 \subset \ldots$ is an increasing sequence of sub-$\sigma$-algebras of $\mathcal{F}$. A random variable $T : \Omega \to \{0, 1, 2, \ldots \} \cup \{\infty\}$ is called a stopping time if for each $n$, the event $\{T \leq n\}$ is in $\mathcal{F}_n$. Let $\mathcal{F}_T$ denote the set of events $A \in \mathcal{F}$ such that for each $n$, $A \cap \{T \leq n\} \in \mathcal{F}_n$. Show that $\mathcal{F}_T$ is a $\sigma$-algebra.

Exercise 3 Suppose $\mu$ is a finite, positive Borel measure on $\mathbb{R}^k$. We will call $\mu$ $d$-dimensional if

$$\lim_{r \to 0} \frac{\log \mu(B(x, r))}{\log r} = d, \text{ a.e.}(\mu).$$

1. Let $\mu$ denote the Cantor measure (the measure generated by the Cantor function). Show that $\mu$ is $d$-dimensional for some $0 < d < 1$ and find $d$.

2. Show that if $\mu$ is absolutely continuous with respect to Lebesgue measure, then $\mu$ is a $k$-dimensional measure.

3. Show that if $\mu(\mathbb{R}^k \setminus V) = 0$ for a countable set $V$, then $\mu$ is $0$-dimensional.

4. (Harder) Construct examples to show that the converses of the last two statements are false. (You may restrict to $k = 1$.)