Math 312, Autumn 2007
Problem Set 8

Rudin, Chapter 8: 3, 5 (this has many parts!), 12, 15

Exercise 1 *Suppose $X_1, X_2, \ldots$ are independent, identically distributed random varaibles with $\mathbf{P}\{X_j = 0\} < 1$. Let $S_n = X_1 + \cdots + X_n$ and N a positive integer. Let*

$$T = \min\{n : |S_n| \geq N\}.$$

Show that there exist positive numbers c, a such that for all n,

$$\mathbf{P}\{T \geq n\} \leq c\, e^{-an}.$$

Conclude that $\mathbf{E}[T] < \infty$.

Exercise 2 *Prove Jensen's inequality for conditional expectation: suppose $f : \mathbb{R} \to \mathbb{R}$ is convex and X is an integrable random variable such that $f(X)$ is also integrable. Then*

$$\mathcal{E}[f(X) \mid \mathcal{G}] \geq f(\mathcal{E}[X \mid G]).$$

(Hint: One approach starts as follows. Show that for any convex f there is a countable collection of real numbers α_j, β_j such that

$$f(x) = \sup[\alpha_j x + \beta_j]. \quad)$$

Exercise 3 *Suppose that M_n is a martingale with respect to a filtration $\mathcal{F}_n$.*

1. *Show that if $p \geq 1$, $m < n$ and E is $\mathcal{F}_m$-measurable, then*

$$\mathbf{E}\left[|M_n|^p \, 1_E\right] \geq \mathbf{E}\left[|M_m|^p \, 1_E\right].$$

2. *Let*

$$M_n^* = \max\left\{|M_1|, \ldots, |M_n|\right\}.$$

Show that for each integer n and each $\lambda > 0$,

$$\mathbf{P}\left\{M_n^* \geq \lambda\right\} \leq \frac{1}{\lambda}\mathbf{E}\left[|M_n|\right].$$

Hint: consider the set

$$E_j = \{M_{j-1}^* < \lambda, M_j^* \geq \lambda\}.$$

3. *Show that for each $p > 1$, there is a $c_p < \infty$ such that for all martingales,*

$$\mathbf{E}\left[(M_n^*)^p\right] \leq c_p \, \mathbf{E}\left[|M_n|^p\right].$$

(Optional: find the optimal c_p.)

4. *Give an example to show that this is not true for $p = 1$.*