Math 312, Autumn 2007
Final Problem Set

Rudin, Chapter 5, 17; Chapter 6, 11 (you may assume Problem 10 as being given); Chapter 9: 2 (but please change the word characteristic to indicator in the problem!), 6, 8, 19

Exercise 1 *A Hamel basis for a complex vector space V is a collection of vectors $\{v_\alpha\}$ such that each $v \in V$ can be written in a unique way as*

$$v = \sum_\alpha c_\alpha \, v_\alpha,$$

where $c_\alpha \in \mathbb{C}$ and all but a finite *number of the c_α are zero. The existence of Hamel bases can be shown using the axiom of choice. Regardless of whether or not you believe in the axiom of choice, show that if H is a Hilbert space then any Hamel basis is either finite or uncountable.*

Exercise 2 *If f is a measurable function on $\mathbb{R}$, let*

$$I_f = \{1 \le p < \infty; \|f\|_p < \infty\},$$

where we are using Lebesgue measure.

1. *Which subsets of $[1, \infty)$ equal I_f for some f?*
2. *Which subsets of $[1, \infty)$ equal I_f for some f that vanishes on $\{|x| \ge 1\}$?*

Exercise 3 *We have considered primarily bounded linear functionals on a Hilbert space. Let H be a Hilbert space with a countably infinite basis. Prove or disprove the following: every linear map $\Phi : H \to \mathbb{C}$ is bounded. (You may assume axiom of choice.)*

Exercise 4 *Let $X_1, X_2, \ldots$ be independent random variables with mean zero and such that $\mathbf{P}\{|X_j| > 1\} = 0$.*

1. *Suppose that*

$$\sum \mathrm{Var}[X_n] < \infty.$$

Show that with probability one that the limit

$$\sum_{j=1}^{n} X_j \tag{1}$$

converges.

2. *Show that if*

$$\sum \mathrm{Var}[X_n] = \infty,$$

then with probability one the limit in (1) *does not exist.*

Exercise 5 *Let $X_1, X_2, \ldots$ be independent random variables each normal mean zero, variance one. Let*

$$S_n = X_1 + \cdots + X_n, \qquad S_n^* = \max |S_j|; j = 1, 2, \ldots, n.$$

and let

$$Y = \limsup_{n\to\infty} \frac{S_n}{\sqrt{n \log\log n}}, \qquad Y^* = \limsup_{n\to\infty} \frac{S_n^*}{\sqrt{n \log\log n}}.$$

1. *Show that there exist $c_0, c_1 \in [0, \infty]$ such that with probability one $Y = c_0, Y^* = c_1$. (Use the Kolmogorov Zero-One Law).*

2. *Explain why for every $\lambda > 0$ and every integer n,*

$$\mathbf{P}\{S_n^* \geq \lambda\} \leq 2\,\mathbf{P}\{|S_n| \geq \lambda\} \leq 4\,\mathbf{P}\{S_n \geq \lambda\}.$$

3. *Find a $c_2 < \infty$ such that*

$$\sum_{n=3}^{\infty} \mathbf{P}\{S_{2^n}^* \geq c_2\,\sqrt{2^n \log\log 2^n}\} < \infty.$$

4. *Show that $c_1 < \infty$.*

5. *Find a $c_3 > 0$ such that*

$$\sum_{n=3}^{\infty} \mathbf{P}\{S_{2^n} - S_{2^{n-1}} \geq c_3\,\sqrt{2^n \log\log 2^n}\} = \infty.$$

6. *Show that with probability one*

$$\limsup_{n\to\infty} \frac{S_{2^n} - S_{2^{n-1}}}{\sqrt{2^n \log\log 2^n}} \geq c_3.$$

7. *Show that $c_0 > 0$.*

(If you get stuck on parts of the problem, feel free to do other parts assuming previous parts.)