

Math 312, Autumn 2008

Problem Set 1

Reading: Rudin Chapter 1; Notes, Sections 1,2

Rudin: pp 31–32, 1, 5, 7, 9, 12

Exercise 1 Consider the symmetric group S_n , i.e., the set of permutations on $\{1, 2, \dots, n\}$. Consider the probability measure on S_n that assigns measure $1/n!$ to each permutation σ . For each permutation, let X denote the number of fixed points, i.e., the number of j such that $\sigma(j) = j$. Compute $\mathbf{E}[X]$ and $\mathbf{E}[X^2]$. (Hint: $X = X_1 + \dots + X_n$ where X_j is the indicator function of the event $\{\sigma(j) = j\}$.)

Exercise 2 Call I a basic interval in $[0, 1)$ if $I = \emptyset$, or $I = [a, b)$ for some $0 \leq a < b \leq 1$.

1. Let \mathcal{F}_0 be the set of all sets A of the form

$$A = A_1 \cup \dots \cup A_n$$

where n is a positive integer and A_1, \dots, A_n are disjoint basic intervals. Show that \mathcal{F}_0 is an algebra but not a σ -algebra.

2. Show that $\sigma(\mathcal{F}_0)$ is the Borel subsets of $[0, 1)$.
3. Let $F : \mathbb{R} \rightarrow [0, \infty)$ be an increasing (i.e., nondecreasing) function. Define μ on \mathcal{F}_0 by $\mu(\emptyset) = 0$, $\mu([a, b) = F(b-) - F(a-)$ and if $A = A_1 \cup \dots \cup A_n$, with A_1, \dots, A_n disjoint then

$$\mu(A) = \mu(A_1) + \dots + \mu(A_n).$$

Show that μ is well defined and countably additive on \mathcal{F}_0 .

4. We will show that this implies that μ can be extended uniquely to the Borel subsets. Assume this. What is $\mu(\{a\})$?

Exercise 3 Suppose \mathcal{F} is a collection of subsets of a set X containing the empty set, closed under complementation, and satisfying the following: if $A_1, A_2, \dots \in \mathcal{F}$ are disjoint, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.

1. Show that if \mathcal{F} is also an algebra, then \mathcal{F} is a σ -algebra.
2. Does \mathcal{F} have to be an algebra?

Exercise 4 Suppose $f : \mathbb{R} \rightarrow [0, \infty)$ is a continuous function satisfying

$$\int_{-\infty}^{\infty} |x| f(x) dx < \infty.$$

Let

$$F(s) = \int_{-\infty}^{\infty} \sin(sx) f(x) dx.$$

Show that F is differentiable and compute the derivative. (Hint: use the dominated convergence theorem to justify the interchange of derivative and integral.)

Exercise 5 If X is a set, then an outer measure on X is a function μ^* from the set of subsets of X to $[0, \infty]$ satisfying: $\mu^*(\emptyset) = 0$; if $A \subset B$, then $\mu^*(A) \leq \mu^*(B)$; and countable subadditivity, i.e., for all A_1, A_2, \dots ,

$$\mu^* \left[\bigcup_{n=1}^{\infty} A_n \right] \leq \sum_{n=1}^{\infty} \mu^*(A_n).$$

Given an outer measure, let \mathcal{F} denote the collection of all subsets A such that for every $E \subset X$,

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Show that \mathcal{F} is a σ -algebra. (Hint: it is probably easiest to first show it is an algebra and then to show it is closed under disjoint countable unions.)