**Math 312, Autumn 2008**
**Problem Set 3**

**Reading.** Rudin, Chapter 3; Probability notes, Sections 3 and 4.

**Problems Rudin**: Chapter 3: 4,5,9,11,16,23,25
(Note: Problem 5 is really a problem about random variables, and you might want to "translate" the problem into probability notation.)

**Probability Notes**: Exercise 4.1.

**Exercise 1** *Let $X$ be a random variable with $\mathbf{E}[|X|] < \infty$. Let $X_n = X\, 1\{|X_n| \leq n\}$. Show that if $\alpha > 1$,*

$$\lim_{n\to\infty} \frac{\mathbf{E}[|X_n|^\alpha]}{n^{\alpha-1}} = 0.$$

**Exercise 2** *Suppose $X_1, X_2, \ldots$ are independent random variables each with mean zero. Let*

$$S_n = X_1 + \cdots + X_n.$$

*Prove that for every $\lambda > 0$,*

$$\mathbf{P}\left\{\omega : \max_{1\leq j\leq n} S_j(\omega) \geq \lambda\right\} \leq \frac{\mathbf{E}[S_n^2]}{\lambda^2}.$$

*(Hint: Let $T = \min\{j : S_j \geq \lambda\}$, let $A_j$ be the event $\{T = j\}$. Show that for every $j \leq n$,*

$$\mathbf{E}\left[S_n^2\, 1_{A_j}\right] \geq \mathbf{E}\left[S_j^2\, 1_{A_j}\right] \geq \lambda^2\, \mathbf{P}(A_j). \quad )$$

**Exercise 3** *Suppose $X_1, X_2, \ldots$ are independent, identically distributed random variables with mean zero and let $\hat{X}_n = X_n\, 1\{|X_n| \leq n\}$,*

$$S_n = \hat{X}_1 + \cdots + \hat{X}_n.$$

*The goal of this exercise is to show*

$$\sum_{n=1}^{\infty} 2^{-2n}\, \mathrm{Var}[S_{2^n}] < \infty,$$

*by proving the following steps.*

1. *For each $n$,*

$$\mathrm{Var}[S_{2^n}] \leq 2^n\, \mathbf{E}\left[X_1^2\, 1\{|X_1| \leq 2^n\}\right].$$

2. 

$$\sum_{n=1}^{\infty} 2^{-2n}\, \mathrm{Var}[S_{2^n}] \leq \mathbf{E}\left[X_1^2 \sum_{n=1}^{\infty} 2^{-n}\, 1\{|X_1| \leq 2^n\}\right].$$

3. 

$$\sum_{n=1}^{\infty} 2^{-n}\, 1\{|X_1| \leq 2^n\} \leq \frac{1}{|X_1|}.$$