Math 312, Autumn 2008
FINAL PROBLEM SET

This problem set is due at 5:00 pm on Thursday, December 4.

You may not consult with anyone else when working on these problems. You may consult inanimate sources. You may not quote results from sources other than Rudin, probability notes, or class lectures.

IMPORTANT INSTRUCTIONS: You are to hand in the four Rudin problems, Exercise 6, and your choice of any three of Exercises 1–5.

Rudin: Chapter 8: #14. Chapter 9:#6,#8, #19.

Exercise 1. *An extreme point of a convex set V in a vector space is a point x such that if $\lambda y + (1-\lambda)z = x$ for some $y, z \in V$ and $\lambda \in (0,1)$, then $y = z = x$. Find the extreme points of the following sets.*

- *The extreme points of the unit sphere in $L^1[0,1]$.*
- *The extreme points of the unit sphere in $l^p, 1 < p < \infty$.*
- *The extreme points of $L^\infty[0,1]$.*
- *The extreme points of $C(V)$ (under the sup norm) where V is the closed unit ball in $\mathbb{R}^k$.*

Exercise 2. *Suppose $f \geq 0$ is an L^1 function on $[0,1]$. Let*

$$r_n = \int_0^1 x^n\, f(x)\, dx.$$

Prove that if $0 \leq k \leq n$ are integers,

$$r_k\, r_{n-k} \leq r_0\, r_n.$$

Exercise 3. *Let X denote the Banach space of continuous function on $[0,1]$ with the sup norm. Let X_0 denote the subset of X consisting of Lipschitz functions, i.e., functions f for which there exists $K < \infty$ such that*

$$|f(x) - f(y)| \leq K|x-y| \text{ for all } x, y \in [0,1].$$

Show that X_0 is of the first category (see Rudin, p.98 for definition).

Exercise 4. *Suppose H is a Hilbert space and $B : H \times H \to \mathbb{C}$ is a bilinear form, i.e.,*

$$B(x,y) = \overline{B(y,x)}$$

$$B(a_1x_1 + a_2x_2, y) = a_1B(x_1,y_1) + a_2B(x_2,y_2).$$

Suppose also that there exist constants C_1, C_2 such that $|B(x,y)| \leq C_2\, \|x\|\, \|y\|$ and $|B(x,x)| \geq C_1\, \|x\|^2$.

- *Show that there is a bounded linear transformation $T : H \to H$ such that for all x, y,*

$$B(x, y) = \langle Tx, y \rangle.$$

- *Show that T is one-to-one and onto.*

Exercise 5. *True or false (give proof or counterexample): If $U \subset \mathbb{R}^k$ is a bounded open set, then the measure of U is the same as the measure of the closure of U.*

Exercise 6. *(This problem is multi-part and will be considered as more than one problem. Feel free to hand in partial solutions to get partial credit.) Let p_k be a sequence of nonnegative numbers with*

$$p_0 > 0, \quad \sum_{k=0}^{\infty} p_k = 1, \quad \mu = \sum_{k=0}^{\infty} k\, p_k, \quad \sum_{k=0}^{\infty} k^2\, p_k < \infty.$$

Suppose that at time 0 a population has 1 individual and the population evolves in the following simple way:

- *At each integer time n, each individual at time $n - 1$ produces a random number of offspring. The probability that a given individual has exactly k offspring is p_k. All the individuals reproduce independently. After reproduction, the individual dies.*

Hence at time n the only members of the population are the children of members at time $n - 1$.

1. *Show that $M_n = \mu^{-n} X_n$ is a martingale.*

2. *Show that if $\mu \leq 1$, then with probability one $X_n = 0$ for all n sufficiently large.*

3. *Show that if $\mu > 1$, there is a K such that for all n,*

$$\mathbf{E}[M_n^2] \leq K.$$

4. *Show that the last statement is false if $\mu \leq 1$ and $p_0 > 0$.*

5. *Show that with probability one for all μ, there is a random variable M_∞ such that with probability one*

$$\lim_{n \to \infty} M_n = M_\infty.$$

6. *Under what conditions is $\mathbf{E}[M_\infty] = 1$?*

7. *Let $q = \mathbf{P}\{M_\infty = 0\}$. Show that q is the smallest nonnegative solution to the equation*

$$q = \sum_{k=0}^{\infty} p_k\, q^k.$$