Math 312, Autumn 2009
Problem Set 1

Reading: Rudin Chapter 1; Notes, Sections 1,2
Rudin: pp 31–32, 1, 3, 7, 12

Exercise 1 *Consider the symmetric group S_n, i.e., the set of permutations on $\{1, 2, \ldots, n\}$. Consider the probability measure on S_n that assigns measure $1/n!$ to each permutation σ. For each permutation, let X denote the number of fixed points, i.e., the number of j such that $\sigma(j) = j$. Compute $\mathbf{E}[X]$ and $\mathbf{E}[X^2]$. (Hint: $X = X_1 + \cdots + X_n$ where X_j is the indicator function of the event $\{\sigma(j) = j\}$.)*

Exercise 2 *Consider the symmetric group S_{2n}. Let q_{2n} be the probability that a permutation chosen at random from S_{2n} has a cycle of length at least $n + 1$. Find constants c_1, c_2 such that the following holds for all positive integers n,*

$$\left| q_{2n} - \log 2 - \frac{c_1}{n} \right| \leq \frac{c_2}{n^2}. \tag{1}$$

You may wish to follow this outline.

1. *For each k, find the probability that a particular integer j is in a cycle of length k.*
2. *For a fixed k, find the expected number of integers that are in a cycle of length k.*
3. *Use this to give an exact expression for q_{2n}.*
4. *Do the (advanced) calculus exercise to derive (1).*

Exercise 3 *Call I a* basic interval *in $[0, 1)$ if $I = \emptyset$, or $I = [a, b)$ for some $0 \leq a < b \leq 1$.*

1. *Let $\mathcal{F}_0$ be the set of all sets A of the form*
$$A = A_1 \cup \cdots \cup A_n$$
where n is a positive integer and $A_1, \ldots, A_n$ are disjoint basic intervals. Show that $\mathcal{F}_0$ is an algebra but not a σ-algebra.
2. *Show that $\sigma(\mathcal{F}_0)$ is the Borel subsets of $[0, 1)$.*
3. *Let $F : \mathbb{R} \to [0, \infty)$ be an increasing (i.e., nondecreasing) function. Define μ on $\mathcal{F}_0$ by $\mu(\emptyset) = 0$, $\mu([a, b) = F(b-) - F(a-)$ and if $A = A_1 \cup \cdots \cup A_n$, with $A_1, \ldots, A_n$ disjoint then*
$$\mu(A) = \mu(A_1) + \cdots + \mu(A_n).$$
Show that μ is well defined and countably additive on $\mathcal{F}_0$.

4. *We will show that this implies that μ can be extended uniquely to the Borel subsets. Assume this. What is $\mu(\{a\})$?*

Exercise 4 *Suppose $\mathcal{F}$ is a collection of subsets of a set X containing the empty set, closed under complementation, and satisfying the following: if $A_1, A_2, \ldots \in \mathcal{F}$ are disjoint, then $\cup_{n=1}^{\infty} A_n \in \mathcal{F}$.*

1. *Show that if $\mathcal{F}$ is also an algebra, then $\mathcal{F}$ is a σ-algebra.*

2. *Does $\mathcal{F}$ have to be an algebra?*

Exercise 5 *If X is a set, then an outer measure on X is a function μ^* from the set of subsets of X to $[0, \infty]$ satisfying: $\mu^*(\emptyset) = 0$; if $A \subset B$. then $\mu^*(A) \leq \mu^*(B)$; and countable subadditivity, i.e., for all $A_1, A_2, \ldots$,*

$$\mu^* \left[\bigcup_{n=1}^{\infty} A_n \right] \leq \sum_{n=1}^{\infty} \mu^*(A_n).$$

Given an outer measure, let $\mathcal{F}$ denote the collection of all subsets A such that for every $E \subset X$,

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Show that $\mathcal{F}$ is a σ-algebra. (Hint: it is probably easiest to first show it is an algebra and then to show it is closed under disjoint countable unions.)

Exercise 6 *Suppose $f : \mathbb{R} \to [0, \infty)$ is a continuous function satisfying*

$$\int_{-\infty}^{\infty} |x| \, f(x) \, dx < \infty.$$

Let

$$F(s) = \int_{-\infty}^{\infty} e^{isx} \, f(x) \, dx.$$

Use the dominated convergence theorem to justify

$$F'(s) = i \int_{-\infty}^{\infty} x \, e^{isx} \, f(x) \, dx.$$