Math 312, Autumn 2009
Problem Set 2 (due Oct 14)

Reading. Rudin, Chapter 2 (but only quickly for pp 35 – 47)
Exercises: Chapter 2: 5, 7, 15, 24, 25

Exercise 1 *Given an example of an algebra $\mathcal{F}$ on a set X and a countably additive measure μ on $\mathcal{F}$ for which there is more than one extension of μ to $\sigma(\mathcal{F})$ that is a measure. (Recall that this implies that μ is not σ-finite.) In your example, state which extension arises from the outer measure as in the proof of the Carathéodory Extension Theorem.*

Exercise 2 *Does there exist a countably additive probability measure on the Borel subsets of $[0,1)$ such that every open interval gets positive measure and the collection of measurable sets is all subsets of $[0,1)$?*

Exercise 3 *Consider $[0,1]$ with Lebesgue measure as a probability space. Write each $\omega \in [0,1]$ in its dyadic expansion $\omega = \omega_1\omega_2\cdots$, i.e.,*

$$\omega = \sum_{n=1}^{\infty} \frac{\omega_n}{2^n}.$$

This expansion is unique for almost every x (which suffices for this exercise). Define the random variable:

$$Y(\omega) = \sum_{n=1}^{\infty} \frac{2\omega_n}{3^n}.$$

1. *What is the distribution function for Y? Show it is continuous.*
2. *Find a Borel subset B of $[0,1]$ which has zero Lebesgue measure for which $\mathbf{P}\{Y \in B\} = 1$.*

Exercise 4 *Suppose X is a nonnegative random variable and $\alpha > 0$.*

1. *Show that*
$$\mathbf{E}[X^\alpha] = \alpha \int_0^\infty x^{\alpha-1}\,\mathbf{P}\{X \geq x\}\,dx.$$
2. *Show that $\mathbf{E}[X^\alpha] < \infty$ if and only if*
$$\sum_{n=1}^{\infty} n^{\alpha-1}\mathbf{P}\{X \geq n\} < \infty.$$

Exercise 5 *Consider a pertmutation σ chosen at random from the uniform distribution on the symmetric group of n elements. Let*

$$X = \#\{j : \sigma(j) = j\}.$$

1. *Show that*
$$\left|\mathbf{P}\{X = 0\} - \frac{1}{e}\right| \leq \frac{1}{(n+1)!}. \tag{1}$$
(Hint: use inclusion-exclusion to give an exact expression for the probability.)

2. *Show that if k is a positive integer,*
$$\lim_{n\to\infty} \mathbf{P}\{X = k\} = \frac{1}{k!\, e}.$$

3. *A popular science magazine a number of years ago gave a problem to determine*
$$\lim_{n\to\infty} \mathbf{P}\{X = 0\}.$$
The official solution was that the probability was exactly
$$\left(1 - \frac{1}{n}\right)^n$$
and hence had limit $1/e$. Show that the answer above is NOT exactly correct for large n by estimating how close it is to $1/e$.

Exercise 6 *Let $\mathcal{C}$ denote the set of bounded Borel measurable functions $f : [0,1] \to \mathbb{R}$. Show that $\mathcal{C}$ is the smallest set of functions containing all the continuous function on $[0,1]$ that satisifies the following closure property: if $f_n \in \mathcal{C}$ are uniformly bounded and the pointwise limit*

$$f(x) = \lim_{n\to\infty} f_n(x),$$

exists for all $x \in [0,1]$, then $f \in \mathcal{C}$.