Math 312, Autumn 2009
Problem Set 3

Reading. Rudin, Chapter 3; Probability notes, Sections 3 and 4.

Problems Rudin: Chapter 3: 4,5,9,12,16,18,24(a),26
(Note: Problem 5 is really a problem about random variables, and you might want to "translate" the problem into probability notation.)

Probability Notes: Exercise 4.1.

Exercise 1 *Suppose $X_1, X_2, \ldots$ are independent random variables each with mean zero. Let*

$$S_n = X_1 + \cdots + X_n.$$

Prove that for every $\lambda > 0$,

$$\mathbf{P}\left\{\omega : \max_{1\leq j\leq n} S_j(\omega) \geq \lambda\right\} \leq \frac{\mathbf{E}[S_n^2]}{\lambda^2}.$$

(Hint: Let $T = \min\{j : S_j \geq \lambda\}$, let A_j be the event $\{T = j\}$. Show that for every $j \leq n$,

$$\mathbf{E}\left[S_n^2\, 1_{A_j}\right] \geq \mathbf{E}\left[S_j^2\, 1_{A_j}\right] \geq \lambda^2\, \mathbf{P}(A_j). \quad)$$

Exercise 2 *Suppose $X_1, X_2, \ldots$ are independent, identically distributed random variables with mean zero and let $\hat{X}_n = X_n\, 1\{|X_n| \leq n\}$,*

$$S_n = \hat{X}_1 + \cdots + \hat{X}_n.$$

The goal of this exercise is to show

$$\sum_{n=1}^{\infty} 2^{-2n}\, \mathrm{Var}[S_{2^n}] < \infty,$$

by proving the following steps.

1. *For each n,*
$$\mathrm{Var}[S_{2^n}] \leq 2^n\, \mathbf{E}\left[X_1^2\, 1\{|X_1| \leq 2^n\}\right].$$

2. $$\sum_{n=1}^{\infty} 2^{-2n}\, \mathrm{Var}[S_{2^n}] \leq \mathbf{E}\left[X_1^2 \sum_{n=1}^{\infty} 2^{-n}\, 1\{|X_1| \leq 2^n\}\right].$$

3. $$\sum_{n=1}^{\infty} 2^{-n}\, 1\{|X_1| \leq 2^n\} \leq \frac{1}{|X_1|}.$$