Math 312, Autumn 2009
Problem Set 4

Reading: Rudin, Chapter 4.

Rudin, Chapter 4: 1, 2, 6, 7, 9, 10, 13, 14, 17
Probability Notes: 4.5, 4.6

Exercise 1 *Let* $(\Omega, \mathcal{F}, \mathbf{P})$ *be a probability space and let* $H_{\mathcal{F}}$ *denote the real Hilbert space* $L^2(\Omega, \mathcal{F}, \mathbf{P})$ *of square-integrable (real valued) random variables. Suppose* $\mathcal{G} \subset \mathcal{F}$ *is a sub* σ*-algebra, and let* $H_{\mathcal{G}}$ *denote the corresponding Hilbert space of* $\mathcal{G}$*-measurable, square-integrable random variables.*

1. *Show that* $H_{\mathcal{G}}$ *is a closed subspsace of* $H_{\mathcal{F}}$*.*

2. *If* $X \in H_{\mathcal{F}}$ *denote by* $\mathcal{E}(X \mid G)$ *the projection of* X *onto* $H_{\mathcal{G}}$*. Show that for all events* $A \in \mathcal{G}$*,*
$$\mathbf{E}\left[X \, 1_A\right] = \mathbf{E}\left[\mathcal{E}(X \mid \mathcal{G}) \, 1_A\right] \tag{1}$$

3. *Suppose* X *is a random variable in* $(\Omega, F, \mathbf{P})$ *with* $\mathbf{E}[|X|] < \infty$ *(note that we do not assume that* $\mathbf{E}[X^2] < \infty$*). Show that there is an integrable random variable* $\mathcal{E}(X \mid \mathcal{G})$ *that is* $\mathcal{G}$*-measurable and such that* (1) *holds for all* $A \in \mathcal{G}$*. (You may wish to consider* $X \geq 0$ *first. You may not use the Radon-Nikodym theorem.)*

4. *Suppose that* X, Y *are square-integrable random variables and* Y *is* $\mathcal{G}$*-meaasurable. Show that*
$$\mathcal{E}(XY \mid \mathcal{G}) = Y \, \mathcal{E}(X \mid \mathcal{G}).$$

5. *Suppose* $\tilde{\mathcal{G}}$ *is independent of* $\mathcal{G}$ *and* X *is* $\tilde{\mathcal{G}}$*-measurable. What is* $\mathcal{E}(X \mid \mathcal{G})$*?*

6. *Show that*
$$\mathbf{E}\left[\mathcal{E}(X \mid \mathcal{G})^2\right] \leq \mathbf{E}[X^2],$$
where we allow infinity as a possible value for the expectation. Give an example to show that the left-hand side can be finite and the right-hand side infinite.