Math 312, Autumn 2009
Problem Set 6

Reading Rudin, Chapter 8 (through page 171)

Rudin, Chapter 8: 1, 3, 4, 12, 15

Exercise 1 *Suppose that $X_1, X_2, \ldots$ are independent, mean zero, variance one random variables not necessarily identically distributed. Let $Z_n = (X_1 + \cdots + X_n)/\sqrt{n}$. Give an example to show that it is not necessarily true that for all $a < b$,*

$$\lim_{n\to\infty} \mathbf{P}\{a \leq Z_n \leq b\} = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx. \tag{1}$$

(Hint: try a modification of the example in Exercise 4.5 of the probability notes.)

Exercise 2 *Suppose in the last exercise we also assume that there is a $K < \infty$ such that $\mathbf{E}[|X_j|^3] \leq K$ for all j. Let ϕ_j denote the characteristic function of X_j.*

1. *Show that there is a c such that for all j and all $t \in \mathbb{R}$,*

$$\left|\phi_j(t) - \left[1 - \frac{t^2}{2}\right]\right| \leq c\,|t|^3.$$

2. *Show that the central limit theorem holds in this case, i.e., for all $a < b$, (1) holds.*