Exercise 1  Suppose that $X_1, X_2, \ldots$ are independent, mean zero, variance one random variables not necessarily identically distributed. Let $Z_n = (X_1 + \cdots + X_n)/\sqrt{n}$. Give an example to show that it is not necessarily true that for all $a < b$,

$$\lim_{n \to \infty} P\{a \leq Z_n \leq b\} = \int_{a}^{b} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx.$$  

(Hint: try a modification of the example in Exercise 4.5 of the probability notes.)

Exercise 2  Suppose in the last exercise we also assume that there is a $K < \infty$ such that $E[|X_j|^3] \leq K$ for all $j$. Let $\phi_j$ denote the characteristic function of $X_j$.

1. Show that there is a $c$ such that for all $j$ and all $t \in \mathbb{R}$,

$$\left| \phi_j(t) - \left[ 1 - \frac{t^2}{2} \right] \right| \leq c |t|^3.$$  

2. Show that the central limit theorem holds in this case, i.e., for all $a < b$, (1) holds.