Math 312, Autumn 2009
Problem Set 8

Reading. Rudin, Chapter 7 (through p. 150); Probability notes, 6.1.
Rudin, Chapter 7: 1,2,5,6,7,12,15

Exercise 1 *Suppose $X_1, X_2, \ldots$ are independent, identically distributed random varaibles with $\mathbf{P}\{X_j = 0\} < 1$. Let $S_n = X_1 + \cdots + X_n$ and N a positive integer. Let*

$$T = \min\{n : |S_n| \geq N\}.$$

Show that there exist positive numbers c, a such that for all n,

$$\mathbf{P}\{T \geq n\} \leq c\, e^{-an}.$$

Conclude that $\mathbf{E}[T] < \infty$.

Exercise 2 *Suppose there is an urn which at time $n = 0$ has one red and one green ball. At each integer time n the balls in the urn are mixed and one ball is chosen at random. The color of that ball is noted and then it and another ball of the same color are put back in the urn. Let W_n denote the number of red balls at time n. The total number of balls at time n is $n + 2$ and hence the number of green balls is $(n + 2) - W_n$. Let*

$$M_n = \frac{W_n}{n+2}$$

denote the fraction of red balls at time n.

1. *Show that M_n is a martingale.*
2. *Show that for each n, M_n is uniformly distributed over the set*
$$\frac{1}{n+2}, \frac{2}{n+2}, \ldots, \frac{n+1}{n+2}.$$
3. *Suppose $M_n = a < b \leq 1$ for some n and let T be the first time $m > n$ that $M_m \geq b$. Show that*
$$\mathbf{P}\{T < \infty \mid M_n = a\} \leq \frac{a}{b}.$$
4. *Show that there is a random variable M_∞ such that with probability one*
$$\lim_{n\to\infty} M_n = M_\infty.$$
5. *What is the distribution of M_∞?*

Exercise 3 *Suppose μ is a finite, positive Borel measure on $\mathbb{R}^k$. We will call μ d-*dimensional *if*

$$\lim_{r \to 0} \frac{\log \mu(\mathcal{B}(x,r))}{\log r} = d, \quad a.e.(\mu).$$

1. *Let μ denote the Cantor measure (the measure generated by the Cantor function). Show that μ is d-dimensional for some $0 < d < 1$ and find d.*
2. *Show that if μ is absolutely continuous with respect to Lebesgue measure, then μ is a k-dimesional measure.*
3. *Show that if $\mu(\mathbb{R}^k \setminus V) = 0$ for a countable set V, then μ is 0-dimensional.*
4. *(Harder) Construct examples to show that the converses of the last two statements are false. (You may restrict to $k = 1$.)*