Partial Differential Equations

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Contents

Preface

The content of this text grew from the lectures taught by the author in the class on Partial Differential Equations for several years at the University of Chicago.

Some undergraduate textbooks on partial differential equations focus on the more computational aspects of the subject: the computation of analytical solutions of equations and the use of the method of separation of variables. While these topics cannot be completely excluded from a first course on PDE at the undergraduate level, we think that it is most useful to focus on the theory of PDE (sometimes relegated to graduate courses), and less on computations. Analytical solutions to partial differential equations are rarely ever practical. Formulas exist only in specific simplistic scenarios, that start and finish with the usual examples presented in textbooks. Beyond those instances, it is the general theory of PDE what helps us best understand the properties of solutions, and eventually also better design numerical algorithms to compute the value of the solution.

Some alternative textbooks that are commonly used in undergraduate PDE courses across the United States are by W. Strauss [9], and by Pinchover and Rubinstein [6]. A book by S. Salsa [8] features extensive review of applications of PDE and modelling. The author was also influenced by a book by I. Peral Alonso [5], which is unfortunately not available in English. Other more advanced textbooks that cover graduate level topics, as well as introductory material, are the classical book by L.C. Evans [2] and a relatively new ambitious book by R. Choksi [1].

This text aims at providing a concise introduction to Partial Differential Equations at the undergraduate level, accessible without the need of too many prerequisites, but at the same time challenging for students of all backgrounds. Another objective is to provide a good collection of exercises that enrich the material presented in every chapter. Some of the exercises can be challenging.

The prerequisites for the PDE class at the University of Chicago are a full sequence on real analysis, following Rudin's *Principles of Real Analysis* [7] or an equivalent textbook, classes on complex analysis and ODEs. The students are not required to have taken a class on measure theory or functional analysis. However, in practice many students have. The notes are written with the intent of making the material accessible and interesting for everybody. For example, Parseval's identity is given in Section 2.8.4, and a remark explains how it can be used to extend the Fourier transform to all of L^2. This remark would be difficult to appreciate for students without a background in measure theory, but it is not necessary for the rest of the text. Another consequence is that some of the proofs appear slightly hand-wavy as an attempt to make the ideas most accessible to students of all backgrounds. A reader with some further knowledge on measure theory and/or functional analysis should be able to easily recognize connections and ideas to move forward and beyond the initial purpose of the text. For example, in Section 4.1, we show that the solution to the Laplace equation minimizes the value of the Dirichlet energy. There is a comment saying that this minimization problem can be used to prove the existence of solutions using techniques from functional analysis, but we only explain its use as a method to establish the uniqueness of solutions. In Section 6.2, we state the notion of weak and entropy solutions of conservation laws for functions that are piecewise continuous. A student with a strong enough background on measure theory should easily recognize how the same notion applies to functions in L^1, and this is indicated in the text.

Many parts in the analysis of PDE require skilled use of vector calculus techniques. While the students taking PDE are supposed to have a background in multivariable calculus, it is a cumbersome topic for people of any level of expertise. To mitigate the trouble and frustration with the required background material, I included a *Preliminaries* chapter to review the formulas from vector calculus and ODEs that are later used in the course.

Preliminaries

The analysis of partial differential equations involves the use of techinques from vector calculus, as well as basic theorem about the solvability of ordinary differential equations. This preliminary material is usually covered in a standard multivariable calculus class and/or a real analysis sequence.

In this preliminary section we collect the main prerequisites for later chapters. We summarize the notation and results from vector calculus that we use throughout the text. We also recall Picard's theorem about the solvability of ODE's.

0.1 Notation

0.1.1 Partial derivatives

There exist different ways to write partial derivatives. We try to use the notation that makes the formulas easiest to read.

The shortest notation is to use subindexes. That is

$$\begin{aligned} u_x &:= \frac{\partial u}{\partial x}, \\ u_{xx} &:= \frac{\partial^2 u}{\partial x^2}, \\ u_t &:= \frac{\partial u}{\partial t}. \end{aligned}$$

We may use the "∂" notation in some cases to avoid confusion (for example when there is another subindex with a different meaning). We write $\partial_t u$ or $\partial_x u$ to refer to u_t or u_x. When $x \in \mathbb{R}^d$, we write $\partial_i u$ to denote $\partial u/\partial x_i$ for $i = 1, \dots, d$.

0.1.2 Domains and Integrals

When we say a set Ω is a *domain* in $\mathbb{R}^d$, we mean that it is an open subset of $\mathbb{R}^d$. We will often work with connected bounded domains whose boundaries are piecewise smooth. When we write a *smooth domain*, we mean that its boundary $\partial\Omega$ is a smooth surface.

We use the standard notation for integrals

$$\int_\Omega f(x)\, \mathrm{d}x.$$

We may use Riemann or Lebesgue integrals. A basic knowledge of either method of integration would suffice for this introduction to PDEs.

We write $\mathrm{d}S$ for the *differential of surface area*, when integrating on a surface. This is most often used to integrate quantities on the boundary of a piecewise smooth domain.

$$\int_{\partial\Omega} f(x)\, \mathrm{d}S.$$

For a point $x \in \mathbb{R}^d$ and $r > 0$, we write $B_r(x)$ to denote the ball of radius r centered at x. If we write B_r, without specifying its center, we mean the ball centered at the origin. Naturally, B_1 is the ball centered at the origin with unit radius.

We say that a function f is C^k in Ω, when it can be differentiated k times and all of its partial derivatives up to order k are continuous functions.

We write the exponential function $\exp(x) := e^x$.

0.2 Vector calculus

Here, We collect some vector calculus definitions and identities that we will use later on in the text.

A classical reference for vector calculus is the book by Marsden and Tromba [4].

0.2.1 Definitions

Let $\Omega \subset \mathbb{R}^d$ be an open set, $u : \Omega \to \mathbb{R}$ be a scalar function and $F : \Omega \to \mathbb{R}^d$ be a vector field.

The gradient of a scalar function is a vector field whose components are the partial derivatives of u.

$$\nabla u(x) = (\partial_1 u(x), \partial_2 u(x), \dots, \partial_d u).$$

The divergence of a vector field is a scalar function which is the sum of the partial derivatives of the corresponding components of F.

$$\operatorname{div} F = \partial_1 F_1 + \partial_2 F_2 + \cdots + \partial_d F_d.$$

The Laplacian of a scalar function is the sum of the second derivatives in every direction.

$$\Delta u = \partial_{11} u + \cdots + \partial_{dd} u.$$

Directional derivative is the rate of change in certain direction v.

$$D_v u = \lim_{h \to 0} \frac{u(x + hv) - u(x)}{h} = v \cdot \nabla u.$$

The last equality holds provided that $u \in C^1$. Note that for all unit vectors v, the largest directional derivative $D_v u$ is when v is parallel to ∇u.

0.2.2 Basic identities

There are many identities involving the basic differential operators from vector calculus. Here, we collect a few that are used later in the text. The first simple identity relates the Laplacian, divergence and gradient.

$$\Delta u = \operatorname{div} \nabla u.$$

There are other identities that follow directly from the definitions. For example, for any scalar function u and vector field F, we have the following version of the product rule

$$\operatorname{div}(uF) = \nabla u \cdot F + u \operatorname{div} F.$$

The Laplacian is also equivalent to the following formula.

$$\Delta u(x) = \lim_{r \to 0} \frac{2(d+2)}{r^2 |B_r|} \int_{B_r(x)} u(y) - u(x) \,\mathrm{d}y.$$

Here $|B_r|$ stands for the volume of the ball B_r. This last formula is useful to form an intuitive picture of the meaning of the Laplace operator. It says that the Laplacian measures the infinitesimal deviation of the value of the function at one point, from the average of its neighboring values. It also shows naturally that the Laplacian is invariant by translation and rotation.

0.2.3 Change of variables

Let $u : \mathbb{R}^d \to \mathbb{R}$ and $\phi : \mathbb{R}^m \to \mathbb{R}^d$ be two C^1 functions. Their composition $u \circ \phi : \mathbb{R}^m \to \mathbb{R}$ is also C^1. Its partial derivatives satisfy the *chain rule* formula

$$\partial_i [u \circ \phi](x) = \sum_{j=1}^{d} \partial_j u(\phi(x)) \partial_i \phi_j(x).$$

Using matrix notation, this writes as

$$D[u \circ \phi](x) = Du(\phi(x)) \cdot D\phi(x). \qquad \text{Here } D\phi \text{ is the matrix } [D\phi]_{ij} = \{\partial_j \phi_i\}.$$

When $\phi : \Omega \to \mathbb{R}^d$ and $\Omega \subset \mathbb{R}^d$, the change of variables formula is

$$\int_{\phi(\Omega)} u(y) \,\mathrm{d}y = \int_\Omega u(\phi(x)) |\det D\phi(x)| \,\mathrm{d}x.$$

We say $|\det D\phi(x)|$ is the *Jacobian* of the change of variables.

0.3 Integration theorems

The *divergence theorem*, also known as Gauss' theorem, relates the divergence of a vector field F inside a set Ω with its flow across the boundary $\partial\Omega$. It can be understood as a multidimensional version of the fundamental theorem of calculus.

Theorem 0.3.1 (Divergence theorem)**.** *Let F be a C^1 vector field and Ω be a subset of $\mathbb{R}^d$ with a piecewise C^1 boundary (the boundary is a union of finitely many C^1 patches). Then,*

$$\int_\Omega \operatorname{div} F \,\mathrm{d}x = \int_{\partial\Omega} F \cdot \nu \,\mathrm{d}S.$$

Here ν is the unit normal vector which points outwards from $\partial\Omega$. The usual differential of volume in Ω is written $\mathrm{d}x$ and the differential of area on $\partial\Omega$ is $\mathrm{d}S$.

If we apply the divergence theorem to $F = \nabla u$, then we obtain the following corollary.

Corollary 0.3.2. *Let u be a C^2 function and Ω be a subset of $\mathbb{R}^d$ with a Lipschitz boundary. Then,*

$$\int_\Omega \Delta u \,\mathrm{d}x = \int_{\partial\Omega} \nabla u \cdot \nu \,\mathrm{d}S.$$

We will often use the notation $u_\nu = \nu \cdot \nabla u$, so the previous formula becomes

$$\int_\Omega \Delta u \,\mathrm{d}x = \int_{\partial\Omega} u_\nu \,\mathrm{d}S.$$

The following identities follow from applying the divergence theorem to $u\nabla u$ and $u\nabla v - v\nabla u$ respectively. They are often useful and are known by the name *Green identities.*

$$\int_\Omega |\nabla u|^2 + u\Delta u \,\mathrm{d}x = \int_{\partial\Omega} u u_\nu \,\mathrm{d}S,$$
$$\int_\Omega u\Delta v - v\Delta u \,\mathrm{d}x = \int_{\partial\Omega} u v_\nu - v u_\nu \,\mathrm{d}S.$$

The divergence theorem is one of the fundamental integration theorems in vector calculus, together with Green's theorem and Stokes theorem.

0.4 Uniform limits, differentiation and integration

We recall the notion of *uniform convergence*.

Definition 0.4.1. *Given a sequence of functions $f_i : \Omega \to \mathbb{R}$, we say that it converges to f* uniformly *if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ so that whenever $i > N$, $|f_i(x) - f(x)| < \varepsilon$ for all $x \in \Omega$.*

It is important in Definition 0.4.1 that the value of N does not depend on the point $x \in \Omega$.

One remarkable property of uniform convergence is that it preserves continuity.

Proposition 0.4.2. *If $\{f_i\}$ is a sequence of continuous functions that converges uniformly to a function f, then f is also continuous.*

Uniform convergence is equivalent to convergence with respect to the *sup-norm*. That is, for any given continuous function $f : \Omega \to \mathbb{R}$, we define

$$\|f\|_{C(\Omega)} = \sup_{x\in\Omega} |f(x)|.$$

The space of continuous functions $C(\Omega)$ turns out to be a complete metric space with the distance given by $d(f,g) = \|f - g\|_{C(\Omega)}$. Uniform convergence is precisely the notion of convergence that corresponds to this metric.

Another remarkable property of uniform convergence is that it allows us to switch limits with integrals, provided that the domain of integration is bounded.

Proposition 0.4.3. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain of integration. Let $\{f_i\}$ be a sequence of functions on Ω that converge uniformly to f. Then*

$$\lim_{i\to\infty} \int_\Omega f_i(x)\, \mathrm{d}x = \int_\Omega f(x)\, \mathrm{d}x.$$

The result of Proposition 0.4.3 would not necessarily hold true when the sequence f_i converges to f point-wise but not uniformly.

The situation with uniform convergence and differentiation is trickier. It is not necessarily true that if f_i converges uniformly to f, then $\partial_x f_i$ will converge to $\partial_x f$. A weaker statement is true: if f_i converges uniformly to some function f **and** $\partial_x f_i$ converges uniformly to another function g, then $\partial_x f = g$.

The following proposition will be used repeatedly thoughout the book.

Proposition 0.4.4. *Let us consider a function $f : [0,T] \times \Omega \to \mathbb{R}$. We write $f(t,x)$, where $t \in [0,T]$ and $x \in \Omega$. Let us assume that Ω is bounded, f is differentiable, and its time derivative is continuous and bounded, then*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega f(t,x)\, \mathrm{d}x = \int_\Omega \partial_t f(t,x)\, \mathrm{d}x.$$

Proposition 0.4.4 can be proved as a consequence of Proposition 0.4.3 applied to the incremental quotients $\Delta_h f(t,x) := (f(t+h,x) - f(t,x))/h$. Indeed, when f has continuous and bounded derivatives, one can see that $\Delta_h f$ converges to $\partial_t f$ uniformly on any compact set.

Those familiar with Lebesgue integration should note that there are more general limit theorems than Proposition 0.4.3, and they correspondigly imply more general conditions under which the identity in Proposition 0.4.4 holds as well.

0.5 Ordinary differential equations

The most fundamental result regarding the solvability of ordinary differential equations is Picard's theorem.

Theorem 0.5.1 (Picard)**.** *Let $F : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ be a differentiable function such that $\nabla_x F(t,x)$ is bounded. Then, the equation*

$$X'(t) = F(t, X(t)),$$
$$X(0) = X_0,$$

has a unique solution $X : \mathbb{R} \to \mathbb{R}^d$ for any value of $X_0 \in \mathbb{R}^d$.

Solutions to ODEs, like in Theorem 0.5.1, depend continuously on the initial data X_0. This is quantified in the following way. Assume that $|\nabla_x F| \leq C$, and we have two solutions $X(t)$ and $Y(t)$ of the same ODE but with different initial conditions.

$$X'(t) = F(t, X(t)), \qquad X(0) = X_0,$$
$$Y'(t) = F(t, Y(t)), \qquad Y(0) = Y_0.$$

Then, we can estimate their distance $|X(t) - Y(t)|$, for any $t > 0$, using *Gronwall's inequality*:

$$|X(t) - Y(t)| \leq e^{Ct}|X_0 - Y_0|.$$

In practice, it is common to have ODEs of the form $X'(t) = F(t, X(t))$, where F is a C^1 function but $|\nabla_x F|$ is not globally bounded in $\mathbb{R}^d$. In that case, we can only ensure that the solution $X(t)$ of the initial value problem exists for some period of time $(-T_1, T_2)$ for some $T_1, T_2 > 0$. The same situation holds if $F(t, x)$ is defined in some subdomain $x \in \Omega \subset \mathbb{R}^d$.

0.6 Exercises

Exercise 0.1. *Let $u : \Omega \to \mathbb{R}$ where $\Omega \subset \mathbb{R}^d$. Let $x \in \Omega$. Prove the following formula for the Laplacian*

$$\Delta u(x) = \lim_{r \to 0} \frac{2(d+2)}{r^2 |B_r|} \int_{B_r(x)} u(y) - u(x) \, \mathrm{d}y.$$

Hint. Use Taylor's expansion.

Exercise 0.2. *Prove the following formula for the divergence of a differentiable vector field $F : \mathbb{R}^d \to \mathbb{R}^d$.*

$$\operatorname{div} F(x) = \frac{1}{|B_1|} \lim_{r \to 0} \int_{\partial B_1} F(x + r\sigma) \cdot \sigma \, \mathrm{d}S(\sigma)$$

Exercise 0.3. *Let Ω be any open set in $\mathbb{R}^d$ with a piecewise smooth boundary. We write $\partial\Omega$ to denote the boundary of Ω and ν is the unit normal vector pointing outwards. We write $\mathrm{d}S$ to denote the differential of surface. Prove the following two identities.*

$$\int_\Omega |\nabla u|^2 + u\Delta u \, \mathrm{d}x = \int_{\partial\Omega} u u_\nu \, \mathrm{d}S,$$
$$\int_\Omega u\Delta v - v\Delta u \, \mathrm{d}x = \int_{\partial\Omega} u v_\nu - v u_\nu \, \mathrm{d}S.$$

Exercise 0.4. *Let $u : \Omega \to \mathbb{R}$ be a C^2 function and $g : \Omega \to \mathbb{R}$ be continuous. Prove that $\Delta u = g$ if and only if for any C^2 function $\varphi : \Omega \to \mathbb{R}$ so that $\varphi = 0$ on $\partial\Omega$, we have*

$$\int_\Omega \nabla u \cdot \nabla \varphi \, \mathrm{d}x = -\int_\Omega g(x)\varphi(x) \, \mathrm{d}x.$$

Exercise 0.5. *You know that in 2D we can use polar coordinates $(x_1, x_2) = (r\cos\theta, r\sin\theta)$ and then the integration formulas become*

$$\int_{B_R} f(x) \, \mathrm{d}x = \int_0^R \int_0^{2\pi} f(r\cos\theta, r\sin\theta) \; r \, \mathrm{d}\theta \, \mathrm{d}r.$$

Also, in 3D we can use spherical coordinates $(x_1, x_2, x_3) = (r\cos\psi, r\sin\psi\cos\theta, r\sin\psi\sin\theta)$ and then the integration formulas become

$$\int_{B_R} f(x) \, \mathrm{d}x = \int_0^R \int_0^{2\pi} \int_0^\pi f(r\cos\psi, r\sin\psi\cos\theta, r\sin\psi\sin\theta) \; r^2 \sin\psi \, \mathrm{d}\psi \, \mathrm{d}\theta \, \mathrm{d}r.$$

The purpose of this question is to develop an n-dimensional version of these formulas. We will call it polar coordinates in $\mathbb{R}^d$.

We write any point $x \in \mathbb{R}^d$ using its distance to the origin $r \in [0,\infty)$ and the angular variables $\theta \in [0,2\pi)$, and $\psi_1,\dots,\psi_{d-2} \in [0,\pi]$. Any point is written as

$$\begin{aligned}
x_1 &= r\cos(\psi_{d-2}),\\
x_2 &= r\sin(\psi_{d-2})\cos(\psi_{d-3}),\\
&\dots\\
x_k &= r\sin(\psi_{d-2})\sin(\psi_{d-3})\dots\sin(\psi_{d-k})\cos(\psi_{d-1-k}),\\
&\dots\\
x_{d-1} &= r\sin(\psi_{d-2})\sin(\psi_{d-3})\dots\sin(\psi_1)\cos(\theta),\\
x_d &= r\sin(\psi_{d-2})\sin(\psi_{d-3})\dots\sin(\psi_1)\sin(\theta).
\end{aligned}$$

(a). Prove that with the formula above we have $|x| = r$.

(b). Prove the change of variables formula

$$\int_{B_R} f \,\mathrm{d}x = \int_0^R \int_0^{2\pi} \int_0^{\pi} \dots \int_0^{\pi} f\ r^{d-1}\sin(\psi_{d-2})^{d-2}\sin(\psi_{d-3})^{d-3}\dots\sin(\psi_1)\,\mathrm{d}\psi_{d-2}\dots\,\mathrm{d}\psi_1\,\mathrm{d}\theta\,\mathrm{d}r.$$

Hint. The solution to this problem amounts to compute the Jacobian of the change of variables

$$(r,\theta,\psi_1,\dots,\psi_{d-2}) \mapsto x.$$

Let $J_d(r,\theta,\psi_1,\dots,\psi_{d-2})$ be this Jacobian. You may expand the corresponding determinant along its first row to obtain the recurrence relation $J_d(r,\theta,\psi_1,\dots,\psi_{d-2}) = r\sin(\psi_{d-2})^{d-2}J_{d-1}(r,\theta,\psi_1,\dots,\psi_{d-3})$. Then, you can verify the general formula by induction.
Note. If we write $\sigma = x/|x|$, then σ is a unit vector on the sphere ∂B_1. We can write the integral in the previous question in the more compact form

$$\int_{B_R} f(x)\,\mathrm{d}x = \int_0^R \int_{\partial B_1} f(r\sigma)\ r^{d-1}\,\mathrm{d}S(\sigma)\,\mathrm{d}r. \tag{1}$$

Here $\sigma_i = x_i/r$ depends only on the angular variables ψ_i and θ. The expression $\mathrm{d}S(\sigma)$ denotes the differential of surface on the unit sphere ∂B_1. It is indeed the same as the expression above

$$\mathrm{d}S(\sigma) = \sin(\psi_1)\sin(\psi_2)^2\dots\sin(\psi_{d-2})^{d-2}\,\mathrm{d}\psi_1\dots\,\mathrm{d}\psi_{d-2}\,\mathrm{d}\theta.$$

In practice, the expression for the integral in polar coordinates we use for our computations is the cleaner one given in (1).

Exercise 0.6. *Let u be a function as in the previous question. Let x, r, ψ_i and θ have the same meaning. Prove that the following formula holds to express the Laplacian of u in terms of polar coordinates (the left hand side is supposed to be in rectangular coordinates and the right hand side is in polar coordinates).*

$$\Delta u(x) = \frac{\partial^2 u}{\partial r^2} + \frac{d-1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\Delta_S u,$$

where $\Delta_S u$ is some combination of derivatives and second derivatives of u with respect to the angular variables ψ_i and θ. There should be no derivative with respect to r in $\Delta_S u$.

Hint. You have three options.

Brute force: Apply the chain rule to the transformation $x \mapsto (r,\theta,\psi_1,\dots)$ and try to organize the terms as well as you can.

Slicker alternative: Observe that after some angular change of variables, it is enough to prove the formula for $\theta = 0$ and $\psi_i = \pi/2$. Then apply the chain rule at this point.

Even better alternative: Let $g = \Delta u$. Using Question 0.4, write the integral in polar coordinates to get

$$\int_0^R \int_{\partial B_1} (\partial_r u \partial_r \varphi + r^{-2} \nabla_\sigma u \cdot \nabla_\sigma \varphi) r^{d-1} \, \mathrm{d}S(\sigma) \, \mathrm{d}r = -\int_0^R \int_{\partial B_1} g(r\sigma)\varphi(r\sigma) r^{d-1} \, \mathrm{d}S(\sigma) \, \mathrm{d}r.$$

Integrate by parts on the left-hand side. Here, $r^{-1}\nabla_\sigma u$ is the part of ∇u that is tangent to the sphere ∂B_r. Observe that $\nabla_\sigma u$ only involves derivatives of u with respect to the angular variables.

Note. The operator $\Delta_S u$ is the Laplace-Beltrami operator on the sphere ∂B_1.

Exercise 0.7. *Let $F : \mathbb{R}^d \to \mathbb{R}^d$ be any C^1 vector field. Assume that there is some $R > 0$ so that $F(x) = 0$ whenever $|x| > R$. Prove that*

$$\int_{\mathbb{R}^d} \operatorname{div} F(x) \, \mathrm{d}x = 0.$$

Exercise 0.8. *Let $u : \mathbb{R}^d \to \mathbb{R}$ be any C^2 function. Assume that there is some $R > 0$ so that $u(x) = 0$ whenever $|x| > R$. Prove that*

$$\int_{\mathbb{R}^d} \Delta u(x) \, \mathrm{d}x = 0.$$

Exercise 0.9. *Let B_R be a sphere centered at the origin. Prove that for any C^2 function $u : \mathbb{R}^d \to \mathbb{R}$*

$$\int_{\partial B_R} \Delta_S u \, \mathrm{d}S = 0.$$

Exercise 0.10. *Let $f_n : \Omega \to \mathbb{R}$ be a bounded sequence of functions on a bounded domain of integration Ω (which is not a closed set). Assume that for any compact set $K \subset \Omega$, f_n converges to f uniformly on K. Prove that*

$$\lim_{n\to\infty} \int_\Omega f_n(x) \, \mathrm{d}x = \int_\Omega f(x) \, \mathrm{d}x.$$

Note. An example of a sequence as in the previous question would be $f_n(x) = x^n$ in the open set $\Omega = (0, 1)$. In that case $f_n \to 0$ uniformly on any closed subinterval $[a, b]$ with $0 < a < b < 1$, however, it does not converge uniformly on the whole domain $(0, 1)$.

Chapter 1

Introduction to PDEs

1.1 Introduction.

A partial differential equation (PDE for short) is a relationship between partial derivatives of a function that is typically unknown. Generally, we have an equation of the form

$$F(x, u(x), Du(x), D^2u(x), \dots) = 0 \text{ in some domain } \Omega.$$

The equation is accompanied by some extra piece of data about the function u. Usually, the value of u is prescribed on some part of the boundary $\partial\Omega$, and we want to identify the function u that solves this equation.

There is a convenient solvability result for ODEs known as *Picard's theorem*. It says that every ODE has a solution at least for a short time. There is no result of that level of generality for partial differential equations. They need to be studied largely on a case by case basis. There are some subclasses of equations which share some characteristics. The following four basic model equations represent four different types of PDEs.

- **The transport equation**. Given a constant $c \in \mathbb{R}^d$, we look for a function $u(t, x)$ that solves

  $$u_t + c \cdot \nabla u = 0,$$

  with a given initial value $u(0, x) = u_0(x)$.

  In this case, the solution is simply $u(t, x) = u_0(x - tc)$. The transport equation is the easiest first order equation, and serves as a starting point for their study.

- **The heat equation**. For some open domain $\Omega \subset \mathbb{R}^d$, we look for a function $u : [0, \infty) \times \overline{\Omega} \to \mathbb{R}$ solving

  $$u_t = \Delta u \text{ in } (0, \infty) \times \Omega,$$

  with some initial value $u(0, x) = u_0(x)$ and some boundary condition that may consist of prescribing either the value of u or its normal derivative u_ν on $(0, \infty) \times \partial\Omega$.

  This is the simplest example of the class of equations known as *parabolic*. They model the notion of diffusion. In this case the values of the function u evolve in time towards the average of the neighboring values. As time goes to infinity, we will see the solution u converge to a stationary value. Variants of this model appear in a variety of applications: the evolution of temperature in certain body, the density of a chemical pollutant, population dynamics, the Black-Scholes equation from financial mathematics, etc...

- **The wave equation**. For some open domain $\Omega \subset \mathbb{R}^d$, we look for a function $u : [0, \infty) \times \overline{\Omega} \to \mathbb{R}$ solving

  $$u_{tt} = \Delta u \text{ in } (0, \infty) \times \Omega,$$

with some initial value $u(0,x) = f(x)$ and $u_t(0,x) = g(x)$, plus some boundary condition that may consist of prescribing either the value of u or its normal derivative u_ν on $(0,\infty) \times \partial\Omega$.

This is the simplest example of the class of *hyperbolic equations*. They model the evolution of waves through a medium. Here u is the displacement from an equilibrium position. There is an energy that is conserved in time. If the domain Ω is bounded, the solution will exhibit oscillatory behavior and it will not converge as $t \to \infty$.

- **The Laplace equation**. For some open domain $\Omega \subset \mathbb{R}^d$, we look for a function $u : \overline{\Omega} \to \mathbb{R}$ solving

$$\Delta u = 0,$$

with a prescribed boundary value on $\partial\Omega$.

This is the simplest example of the class of *elliptic equations*. They model the stationary case of parabolic or hyperbolic equations. In this case, it is the limit of the solution of the heat equation as $t \to +\infty$.

Elliptic, parabolic, and hyperbolic equations are classes of PDEs for which we are able to apply methods and obtain results similar to those for the Laplace, heat, and wave equations, respectively. The classification of PDEs is simply this vague concept.

We say a problem is **well-posed** if for any value of the given data, there exists a unique solution and it depends continuously on the data.

A *solution* to a partial differential equation is a function that is smooth enough so that all the derivatives involved in the expression are well defined and satisfy the equation at every point of its domain. This is the classical notion of solution and the one we use *by default* throughout this course. There are a few cases, however, where it makes sense to extend our notion of solution to include irregular functions for which the derivatives may not be well defined everywhere. In that case, the notion of solution is subtle and needs an appropriate definition. While this is an advanced topic that falls mostly outside the scope of these notes, we will discuss a few situations where we observe this phenomenon. For example, we may argue that the formula for the transport equation above makes sense even when u_0 is not differentiable. Further examples that we will discuss in these notes are the subharmonic functions in Chapter 4, and the weak solutions of conservation law equations in Chapter 6.

In some particular cases, for simple equations, in simple domains, we can write a formula to obtain the solution to a PDE. We will see several examples of analytic solutions. We start with the method of separation of variables, which is one of the most traditional methods for computing solutions. Later we will also derive some formulas using the fundamental solution of each equation. However, you must be aware that this can only be done in very simple cases. This first introductory course in PDE is probably the first and also the last place you will ever see an analytical solution to a PDE. Most often, real-world PDEs are not explicitly solvable. We study theory that tells us properties of the solutions of PDEs. We prove theorems about the existence of solutions, their uniqueness, regularity, and asymptotic estimates. Out in the real world, the most practical method to *solve* a PDE is through numerical analysis. In order to draw useful conclusions from PDEs, including the development of successful numerical methods, it is important to understand the equations well and know what to expect for the solutions even when there is no formula to solve them. Because of this, learning the *theory* of PDEs is arguably more important than computations with explicit formulas.

1.2 Derivation of the equations

Here, we sketch a physical derivation of the heat and wave equation. Recall that the Laplace equation represents the stationary case of either one. The equations arise from a variety of situations. The descriptions given below are by no means the only way to come up with them. They are, perhaps, their most traditional motivations from basic principles. Our explanation is admittedly very hand-wavy in an attempt to (over?)simplify the exposition. We do not try to be rigorous in any way here.

1.2.1 The heat equation

Suppose that a certain homogeneous physical body occupies a region Ω in space. Let us say that Ω is an open set in $\mathbb{R}^3$. Let $u(t,x)$ be the temperature at any time t and any point $x \in \Omega$. We want to derive an equation for the evolution of the temperature $u(t,x)$ given its initial value $u(0,x) = u_0$ and prescribing the temperature on the boundary $u(t,x)$ for any $x \in \partial\Omega$. A natural physical law says that heat flows in the opposite direction of the gradient of temperature, i.e., from hotter to colder regions. Because we assume that there are no external sources, the rate of change of the amount of heat in any subset $D \subset \Omega$ should balance with the flow of heat through its boundary. This is expressed in the following equation (modulo some physical constant factors that we neglect):

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_D u(t,x)\, \mathrm{d}x + \int_{\partial D} -\nabla u \cdot n \, \mathrm{d}S = 0.$$

We differentiate inside the integral on the left-hand side and apply the divergence theorem to the right-hand side. We are left with:

$$\int_D u_t(t,x) - \Delta u(t,x)\, \mathrm{d}x = 0.$$

Since the equality above must hold for any subdomain $D \subset \Omega$, the function u must satisfy the heat equation:

$$u_t - \Delta u = 0,$$

at every point $(t,x) \in (0,\infty) \times \Omega$.

1.2.2 The wave equation (in one space dimension)

Let us consider a function $u(t,x)$ that measures the vertical displacement in a vibrating spring (like in a guitar). The variable x is one dimensional and takes values in the interval $[0,1]$. We consider a spring that is attached at its endpoints, so $u(t,0) = u(t,1) = 0$ for any value of t. The initial state of the spring is given by its initial position $u(0,x) = u_0(x)$ and its initial velocity $u_t(t,x) = v_0(x)$. We want to derive the PDE that describes the evolution in time of the function u. Our equation is meant to be accurate only for small vertical displacements of the strings. We neglect the horizontal displacement as well as several nonlinear effects. Under these assumptions, we derive the one dimensional wave equation below.

Let us consider any piece of the string, for $x \in (a,b)$. Assuming that the density of the spring is homogeneous and ignoring any possible horizontal displacement, the total vertical momentum of this piece would be proportional to

$$\int_a^b u_t(t,x)\, \mathrm{d}x.$$

From Newton's law, the rate of change of momentum must be equal to the forces applied to this piece of string. In this case, the only forces are applied from the spring tension at the end points $x = a, b$. They point outwards in the tangential direction to the spring. In this case, the outwards tangential directions are $(-1, -u_x(t,a))$ and $(1, u_x(t,b))$, which add up to $u_x(t,b) - u_x(t,a)$ in the vertical direction. We are left with the equality

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_a^b u_t(t,x)\, \mathrm{d}x = c(u_x(t,b) - u_x(t,a)).$$

We collected all the constant factors involved in the derivation into the constant c (which is always positive, and it equals the square of the speed of wave propagations).

Differentiating the left-hand side inside the integral and applying the fundamental theorem of calculus to the right-hand side, we obtain

$$\int_a^b u_{tt}(t,x)\, \mathrm{d}x = c \int_a^b u_{xx}(t,x)\, \mathrm{d}x.$$

Since the identity above must hold for any subinterval $(a,b) \subset [0,1]$, we obtain:

$$u_{tt} = c u_{xx} \text{ for } t \in (0,\infty) \text{ and } x \in (0,1).$$

Chapter 2

Separation of variables and Fourier series

The method of *separation of variables* is a technique for computing the general solution of a PDE as a superposition of special solutions that are expressed as products of functions depending on single variables. It is one of the most classical method to effectively compute the solution of a PDE, when the geometry of the domain is simple. The best way to understand the method is by analyzing a few examples.

2.1 Separation of variables for the heat equation

In this first example, we want to solve the heat equation in a one-dimensional interval with zero Dirichlet boundary conditions and a given initial value $u_0(x)$.

$$u_t = u_{xx} \quad \text{for all } x \in (0,1), \text{ and } t > 0, \tag{2.1}$$
$$u(t,0) = u(t,1) = 0 \quad \text{for all } t > 0, \tag{2.2}$$
$$u(0,x) = u_0(x) \quad \text{for all } x \in [0,1], \tag{2.3}$$

In the method of separation of variables, we first look for special solutions of the form $u(t,x) = A(t)B(x)$ which satisfy (2.1) and (2.2). We ignore the initial data (2.3) for now.

The way to assure that the Dirichlet condition (2.2) is satisfied, is by setting $B(0) = B(1) = 0$. In order to satisfy the equation (2.1), we obtain the following relation between A and B,

$$A'(t)B(x) = A(t)B''(x).$$

Therefore

$$\frac{A'(t)}{A(t)} = \frac{B''(x)}{B(x)}.$$

Since the left-hand side is independent of x, the quantity in this equality does not depend on x. Further, since the right-hand side is independent of t, the quantity cannot depend on t either and thus it is a constant. Let us call it λ.

We obtained the two ordinary differential equations.

$$B''(x) = \lambda B(x) \text{ with } B(0) = B(1) = 0, \tag{2.4}$$
$$A'(t) = \lambda A(t). \tag{2.5}$$

If we replace A and B by $c^{-1}A$ and cB for any arbitrary factor $c \neq 0$ and their product $A(t)B(x)$ remains the same. So, we can normalize the functions A and B so that $A(0) = 1$, without affecting their product. The solution to (2.5) is clearly $A(t) = \exp(\lambda t)$.

From the ODE for $B(x)$, we observe that $B(x) = a_1 \exp(\sqrt{\lambda}x) + a_2 \exp(-\sqrt{\lambda}x)$ for some constants a_1 and a_2 which have to be chosen to satisfy the boundary condition $B(0) = B(1) = 0$. This gives us a linear system of two equations with two unknowns:

$$\begin{aligned} a_1 + a_2 &= 0, \\ a_1 \exp(\sqrt{\lambda}) + a_2 \exp(-\sqrt{\lambda}) &= 0. \end{aligned}$$

It is clear that $a_1 = a_2 = 0$ is a possible solution of the system above. However, this would make $B(x) = 0$ and we would be obtaining the uninteresting solution $u \equiv 0$. The only way a system of two equations with two unknowns can have more than one solution is if the equations are **not** linearly independent. In this case, it can happen only if one equation is a multiple of the other:

$$\left(\exp(\sqrt{\lambda}), \exp(-\sqrt{\lambda})\right) \parallel (1, 1).$$

In other words, we need to have $\exp(\sqrt{\lambda}) = \exp(-\sqrt{\lambda})$ for a non-zero function B to exist. That means $\exp(2\sqrt{\lambda}) = 1$, therefore $2\sqrt{\lambda} = 2\pi k i$ for some integer k. We obtain the function $B(x)$ which is

$$B(x) = \frac{a}{2i}\left(\exp(\pi k i x) - \exp(-\pi k i x)\right) = a \sin(\pi k x).$$

Note that $\lambda = -\pi^2 k^2$. Therefore, we obtained a particular solution of the heat equation (2.1) and (2.2):

$$u(t, x) = A(t)B(x) = a \sin(\pi k x) \exp(-\pi^2 k^2 t). \tag{2.6}$$

Here a is any number. In our analysis above, it was useful to take complex numbers as parameters, so we may let a be any complex number. If we want the function $u(t, x)$ to take only real values, we take $a \in \mathbb{R}$.

This is the solution that we were looking for only if $u_0(x) = a \sin(\pi k x)$. Therefore, so far we have solved the equation only for a particular type of initial values.

If we add two solutions of the heat equation, that gives us another solution. This observation allows us to add several solutions of the form (2.6) for different values of $k \in \mathbb{N}$ and $a \in \mathbb{R}$ obtain further solutions of the form.

$$u(t, x) = \sum_k a_k \sin(\pi k x) \exp(-\pi^2 k^2 t) \tag{2.7}$$

The sum could have infinitely many terms if we add over all positive integers k and that would still give us a solution of the heat equation if the series converges appropriately (for example if the values of the function together with its first and second derivatives converge uniformly). In this way, we can find a solution to the equation for all initial data u_0 which can be expressed as an infinite sum

$$u_0(x) = \sum_{k=1}^{\infty} a_k \sin(\pi k x). \tag{2.8}$$

Two natural questions arise. The first one is: what functions $u_0 : [0, 1] \to \mathbb{R}$ can be expressed by such a sum?

The quick answer to this first question is that all reasonable functions u_0 can. This is related to the theory of Fourier series. For example every $u_0 \in C^2$, is equal to a sum like (2.8) for some choice of coefficients a_k that converges uniformly. We will prove it in a later section.

The second question is, given an explicit function u_0, how can we obtain the corresponding coefficients a_k? We answer this question in the remaining of this section.

We start by observing the following relation (exercise!) [1]

$$\int_0^1 \sin(k_1 \pi x) \sin(k_2 \pi x)\, \mathrm{d}x = \begin{cases} 1/2 & \text{if } k_1 = k_2, \\ 0 & \text{if } k_1 \neq k_2. \end{cases} \tag{2.9}$$

[1] It is easiest to prove the relationship (2.9) using that $\sin(\pi k x) = \frac{\exp(\pi k x i) - \exp(-\pi k x i)}{2}$

Let us multiply the expression (2.8) times $\sin(\pi n x)$ and integrate. We obtain

$$\int_0^1 u_0(x)\sin(\pi n x)\,\mathrm{d}x = \sum_{k=1}^{\infty} a_k \int_0^1 \sin(\pi k x)\sin(\pi n x)\,\mathrm{d}x = \frac{a_n}{2}. \tag{2.10}$$

The expression (2.10) gives us a practical formula to recover a_n integrating $u_0(x)$ against $\sin(\pi n x)$. The computation is justified provided that the series in (2.8) converges uniformly, so that we can commute integration with summation above.

Combining (2.10) with (2.7), we obtain the formula

$$\begin{aligned} u(t,x) &= \sum_k 2\left(\int_0^1 u_0(y)\sin(\pi k y)\,\mathrm{d}y\right)\sin(\pi k x)\exp(-\pi^2 k^2 t) \\ &= \int_0^1 u_0(y)\left(\sum_k 2\sin(\pi k y)\sin(\pi k x)\exp(-\pi^2 k^2 t)\right)\,\mathrm{d}y \end{aligned}$$

If we call

$$H(t,x,y) := \sum_{k=1}^{\infty} 2\sin(\pi k y)\sin(\pi k x)\exp(-\pi^2 k^2 t), \tag{2.11}$$

the formula reads

$$u(t,x) = \int_0^1 H(t,x,y)u_0(y)\,\mathrm{d}y.$$

We derived this formula under the optimistic assumption that the initial data u_0 can be written as a trigonometric series like (2.8). The formula will be fully justified as soon as we verify that this is indeed possible for every function u_0.

The function $H(t,x,y)$ is given more or less explicitly in (2.11). We call the function H the *heat kernel* in the unit interval $[0,1]$. In a later chapter, we will explicitly compute the heat kernel in the full space $\mathbb{R}^d$.

2.2 The homogeneous Laplace equation in a disc

As a second application of the method of separation of variables, we solve the Laplace equation in a two-dimentional disc.

$$\begin{aligned} \Delta u &= 0 \text{ in } B_1, \\ u &= f \text{ on } \partial B_1. \end{aligned} \tag{2.12}$$

Here, B_1 is the unit disc and ∂B_1 is the unit circle.

The Laplacian, in polar coordinates, has the expression

$$\Delta u = u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2}. \tag{2.13}$$

To start with the method of separation of variables, we temporarily ignore the Dirichlet boundary condition $u = f$ on ∂B_1 and look for a family of solutions of the equation $\Delta u = 0$ in B_1. We try special solutions of the form $u(r,\theta) = A(\theta)B(r)$. Plugging a function of this form in the formula (2.13), we obtain

$$B''(r)A(\theta) + \frac{B'(r)A(\theta)}{r} + \frac{B(r)A''(\theta)}{r^2} = 0.$$

We rearrange the terms by putting everything that depends on r on the right hand side, and everything that depends on θ on the left.

$$\frac{A''(\theta)}{A(\theta)} = -\frac{r^2 B''(r) + rB'(r)}{B(r)}.$$

Since the value of this equality cannot depend on either r or θ, it has to be a constant that we call λ. We have the two ODEs depending on λ.

$$\begin{aligned} A''(\theta) - \lambda A(\theta) &= 0, \\ r^2 B''(r) + rB'(r) + \lambda B(r) &= 0. \end{aligned}$$

The first equation tells us that A must have the form

$$A(\theta) = a_1 e^{\sqrt{\lambda}\theta} + a_2 e^{-\sqrt{\lambda}\theta} \qquad \text{if } \lambda \neq 0,$$

$$\text{or } A(\theta) = a_1 + a_2\theta \qquad \text{if } \lambda = 0.$$

Recall that θ represents an angle, and therefore A must necessarily be periodic of period 2π. That means that $\sqrt{\lambda} = ik$ for some integer k.

$$A(\theta) = a_1 e^{ik\theta} + a_2 e^{-ik\theta}. \tag{2.14}$$

(there is only one term in the case $k = 0$)

Remark 2.2.1. If we want further that $A(\theta) \in \mathbb{R}$ for all θ, it can be seen that this happens when $a_1 = \overline{a_2}$. Equivalently, we can write $A(\theta)$ as a sum of trigonometric functions

$$A(\theta) = b_1 \cos(k\theta) + b_2 \sin(k\theta).$$

Because it is easier to make computations with exponential functions, we will stick with the expression (2.14) and allow complex numbers in the coefficients.

Now that we established that $\sqrt{\lambda} = ik$ for some integer k, we can solve the equation for B. We replace $\lambda = -k^2$ in the ODE and get

$$r^2 B''(r) + rB'(r) - k^2 B(r) = 0.$$

It is not hard to find two linearly independent solutions to this equation just by guessing. If $k = 0$, they are either $B = 1$ or $B(r) = \log(r)$. If $k \neq 0$, they are $B = r^k$ and $B = r^{-k}$. The solution we are looking for $u(r, \theta) = A(\theta)B(r)$ should not be discontinuous at the origin, so we discard the solutions $B(r) = \log r$ or the ones which have a negative power.

Therefore, for any nonnegative integer k, and any parameters $a, b \in \mathbb{C}$, we obtained the family of solutions

$$a\ e^{ik\theta} r^k + b\ e^{-ik\theta} r^k.$$

For any value of $a, b \in \mathbb{C}$, and $k \in \mathbb{N}$, these functions solve the Laplace equation in B_1 (they are harmonic). Each one has a different value on ∂B_1. We intend to write a generic harmonic function in B_1, with an arbitrary boundary value, as an infinite series

$$\sum_{k \in \mathbb{Z}} a_k r^{|k|} e^{ik\theta} \qquad \text{(which ultimately will equal } u(r,\theta)) \tag{2.15}$$

Imagine that a_k is any bounded sequence of complex numbers. That means that $|a_k| \leq M$ for some M. If we fix a value of $r < 1$, the series is majorated by

$$|a_k r^{|k|} e^{ik\theta}| \leq M r^{|k|}$$

In particular, this means that the series converges uniformly in $B_{1-\delta}$ for any $\delta > 0$.

The convergence of the series for $r = 1$ is more delicate. In fact, we can only prove it if $|a_k|$ is a convergent series. In that case we use that for $r \leq 1$,

$$|a_k r^{|k|} e^{ik\theta}| \leq |a_k|$$

Thus, (2.15) will converge uniformly in the full closed disc $\overline{B}_1$ if $\sum |a_k| < +\infty$.

This method of separation of variables succeeds in solving the problem (2.12) if we manage to prove that the series (2.15) converges uniformly and it matches the boundary value so that

$$\sum_{k \in \mathbb{Z}} a_k e^{ik\theta} = f(\theta). \tag{2.16}$$

Finding the right coefficients a_k and proving that they make the equality (2.16) true is the objective of the following sections.

2.3 Fourier coefficients

We want to write a generic 2π-periodic function f as a series like (2.15). This type of series is called *Fourier series*, and the coefficients a_k are the Fourier coefficients of f.

The first thing we do is deriving a formula for the Fourier coefficients a_k in terms of the function f. We observe the following identity

$$\int_0^{2\pi} e^{ik\theta} e^{-in\theta} \, \mathrm{d}\theta = \begin{cases} 2\pi & \text{if } k = n, \\ 0 & \text{if } k \neq n. \end{cases}$$

Thus, we would have

$$\frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{k \in \mathbb{Z}} a_k e^{ik\theta} \right) e^{-in\theta} \, \mathrm{d}\theta = a_n.$$

Because we want the formula (2.16) to hold, we define the Fourier coefficients a_k as

$$a_k := \frac{1}{2\pi} \int_0^{2\pi} f(\theta) e^{-ik\theta} \, \mathrm{d}\theta. \tag{2.17}$$

This is our **definition** of the Fourier coefficients a_k corresponding to f.

Note that if f is bounded by a constant M, we see in the expression (2.17) that $|a_k| \leq M$ for all k. As discussed in the previous section, this would imply that the series (2.15) converges uniformly in $B_{1-\delta}$ for all $\delta > 0$. However, when the function f is smooth, we can obtain an even better upper bound for $|a_k|$.

Let us assume that $f \in C^1$. Then we integrate by parts the expression (2.17) and obtain

$$\begin{aligned} |a_k| &= \frac{1}{2\pi} \left| \int_0^{2\pi} f(\theta) e^{-ik\theta} \, \mathrm{d}\theta \right|, \\ &= \frac{1}{2\pi} \left| \int_0^{2\pi} f'(\theta) \frac{e^{-ik\theta}}{ik} \, \mathrm{d}\theta \right|, \\ &\leq \frac{1}{2\pi |k|} \int_0^{2\pi} |f'(\theta)| \, \mathrm{d}\theta, \end{aligned}$$

The boundary terms in the integration by parts cancel out due to the periodicity of the integrand.

We deduce that $|a_k| \leq M/k$ for $M = \max |f'(\theta)|$. If $f \in C^2$, we obtain an even faster decay for $|a_k|$. Indeed, we can integrate by parts twice.

$$\begin{aligned} |a_k| &= \frac{1}{2\pi} \left| \int_0^{2\pi} f(\theta) e^{-ik\theta} \, \mathrm{d}\theta \right|, \\ &= \frac{1}{2\pi} \left| \int_0^{2\pi} f'(\theta) \frac{e^{-ik\theta}}{ik} \, \mathrm{d}\theta \right|, \\ &= \frac{1}{2\pi} \left| \int_0^{2\pi} f''(\theta) \frac{e^{-ik\theta}}{-k^2} \, \mathrm{d}\theta \right|, \\ &\leq \frac{1}{2\pi k^2} \int_0^{2\pi} |f''(\theta)| \, \mathrm{d}\theta, \end{aligned}$$

Therefore, $|a_k| \leq M/k^2$ for $M = \max |f''(\theta)|$. This is enough to conclude that $\sum |a_k| < +\infty$. We frame this result as the next proposition.

Proposition 2.3.1. *If a 2π periodic function is C^2, then its Fourier coefficients a_k are summable. That means*

$$\sum_{k \in \mathbb{Z}} |a_k| < +\infty.$$

As we explained in the previous section, we now have that the series (2.15) converges in the whole disc B_1. In particular, the series in (2.16) also converges uniformly. We have not shown yet that the value of the series (2.16) indeed equals $f(\theta)$. From our construction, if the equality (2.16) holds, then the coefficients a_k have to be the ones given by the formula (2.17). What we are left to justify is that every function f can be written as a Fourier series as in (2.16).

2.4 The Poisson kernel

We now plug the formula for the Fourier coefficients (2.17) into (2.15) and obtain a simplified expression of the value of the series (2.15) in terms of f.

We have

$$\begin{aligned}\sum_{k\in\mathbb{Z}} a_k r^{|k|} e^{ik\theta} &= \sum_{k\in\mathbb{Z}} \left(\frac{1}{2\pi}\int_0^{2\pi} f(\sigma)e^{-ik\sigma}\,\mathrm{d}\sigma\right) r^{|k|}e^{ik\theta},\\ &= \frac{1}{2\pi}\int_0^{2\pi} f(\sigma)\left(\sum_{k\in\mathbb{Z}} r^{|k|}e^{ik(\theta-\sigma)}\right)\mathrm{d}\sigma,\\ &= \int_0^{2\pi} f(\sigma)P(r,\theta-\sigma)\,\mathrm{d}\sigma.\end{aligned}$$

We define the *Poisson kernel* P by the formula

$$P(r,\theta) := \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} r^{|k|}e^{ik\theta}.$$

Note that from our construction, if $f(\theta) = \sum a_k e^{ik\theta}$, then $\int_0^{2\pi} f(\sigma)P(r,\theta-\sigma)\,\mathrm{d}\sigma = \sum a_k r^{|k|}e^{ik\theta}$. In particular, applying this to the case $f\equiv 1$, we get

$$\int_0^{2\pi} P(r,\sigma)\,\mathrm{d}\sigma = 1 \qquad \text{for any value of } r<1.$$

Let us simplify the formula for P. We have

$$\begin{aligned}P(r,\theta) &= \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} r^{|k|}e^{ik\theta},\\ &= \frac{1}{2\pi}\left(\sum_{k=0}^{\infty} r^k e^{ik\theta} + \sum_{k=1}^{\infty} r^k e^{-ik\theta}\right),\\ &= \frac{1}{2\pi}\left(\sum_{k=0}^{\infty}\left(re^{i\theta}\right)^k + \sum_{k=1}^{\infty}\left(re^{-i\theta}\right)^k\right),\\ &= \frac{1}{2\pi}\left(\frac{1}{1-re^{i\theta}} + \frac{re^{-i\theta}}{1-re^{-i\theta}} = \frac{1-re^{-i\theta}+re^{-i\theta}-r^2}{(1-re^{i\theta})(1-re^{-i\theta})}\right),\\ &= \frac{1}{2\pi}\frac{1-r^2}{1-2r\cos(\theta)+r^2}\end{aligned}$$

Thus, we arrive to the most common expression for the Poisson kernel

$$P(r,\theta) = \frac{1}{2\pi}\frac{1-r^2}{1-2r\cos(\theta)+r^2}.$$

Note that for $r<1$, the numerator $1-r^2>0$ and the denominator $1-2r\cos(\theta)+r^2 \geq 1-2r+r^2 = (1-r)^2>0$. Therefore, $P(r,\theta)>0$ for all $r<1$ and $\theta\in[0,2\pi)$.

If we fix a value of θ, we observe that $\lim_{r\to 1} P(r,\theta)=0$ unless $\theta=0$. However, $P>0$ and its integral in θ stays equal to 1 for all r. This indicates that the whole integral of P is concentrating around $\theta=0$ as $r\to 1$. In other words, $P(r,\theta)=\delta_0(\theta)$ as $r\to 1$ (whatever that means).

Now we will make rigorous sense of the previous paragraph. Let $\delta>0$ be an arbitrarily small number. We observe that $P(r,\theta)$ converges uniformly to zero as $r\to 1$ for $\theta\in[\delta,2\pi-\delta]$. Indeed, for these values of θ,

$$P(r,\theta) = \frac{1}{2\pi}\frac{1-r^2}{1-2r\cos(\theta)+r^2} \leq \frac{1}{2\pi}\frac{1-r^2}{1-2r\cos\delta+r^2},$$

which converges to zero as $r \to 1$.

We are now ready to show that if f is any 2π-periodic continuous function, we have

$$\lim_{r\to 1} \int f(\sigma)P(r,\theta-\sigma)\,\mathrm{d}\sigma = f(\theta) \qquad \text{uniformly.}$$

Since f is continuous in $[0, 2\pi]$, in particular it is bounded by some constant M. Let $\varepsilon > 0$ be arbitrary. The function f must be uniformly continuous (since it is a continuous function in the compact set $[0, 2\pi]$), therefore there must be some small $\delta > 0$ so that $|f(x) - f(y)| < \varepsilon$ if $|x - y| < 2\delta$.

Since the integral of $P(r, \theta)$ with respect to θ is equal to one for all r,

$$\left| \int_0^{2\pi} f(\sigma)P(r,\theta-\sigma)\,\mathrm{d}\sigma - f(\theta) \right| = \left| \int_0^{2\pi} \left(f(\sigma) - f(\theta)\right) P(r,\theta-\sigma)\,\mathrm{d}\sigma \right|,$$

Using the 2π-periodicity of f and an elementary change of variables,

$$= \left| \int_0^{2\pi} \left(f(\theta-\sigma) - f(\theta)\right) P(r,\sigma)\,\mathrm{d}\sigma \right|,$$

now we split the domain of integration into two parts,

$$\leq \left| \int_\delta^{2\pi-\delta} (f(\theta-\sigma) - f(\theta))P(r,\sigma)\,\mathrm{d}\sigma \right| + \left| \int_{-\delta}^{\delta} (f(\theta-\sigma) - f(\theta))P(r,\sigma)\,\mathrm{d}\sigma \right|,$$

we bound the first term using the upper bound M for f and the second term using the modulus of continuity,

$$\leq 2M \int_\delta^{2\pi-\delta} P(r,\sigma)\,\mathrm{d}\sigma + \varepsilon \int_{-\delta}^{\delta} P(r,\sigma)\,\mathrm{d}\sigma,$$

we know that P goes to zero uniformly in $[\delta, 2\pi - \delta]$, so the first integral will converge to zero. We bound the second one by one.

$$\leq 2M \left(\int_\delta^{2\pi-\delta} P(r,\sigma)\,\mathrm{d}\sigma \right) + \varepsilon \to \varepsilon \quad \text{as } r \to 1.$$

Since ε is arbitrary. This computation shows that

$$\lim_{r\to 1} \int_0^{2\pi} f(\sigma)P(r,\theta-\sigma)\,\mathrm{d}\sigma = f(\theta) \qquad \text{uniformly.} \tag{2.18}$$

The Poisson kernel is a harmonic function for $r < 1$. This is evident from its construction, but it can also be verified directly by computing its Laplacian. Therefore, we see that

$$u(r,\theta) = \int_0^{2\pi} f(\sigma)P(r,\theta-\sigma)\,\mathrm{d}\sigma,$$

is harmonic in B_1 and also $\lim_{r\to 1} u(r,\theta) = f(\theta)$ uniformly. Thus, this function u is indeed the solution to our original Dirichlet problem (2.12).

2.5 Convergence of Fourier series

In the previous sections we employed the method of separation of variables to derive a formula for the solution of the Dirichlet problem (2.12). The method would succeed provided that the boundary condition

f can be expressed as a Fourier series (2.16). Under this assumption, we derived a formula for the solution using the Poisson kernel. A posteriori, we showed that the formula we obtained indeed solves the problem 2.12. We can argue now that this implies that the expression of a generic function f as a Fourier series (2.16) must be true.

We enclose this fundamental result about Fourier series in the following theorem. It is an important result by itself. It has plenty of applications going beyond the solvability of the Dirichlet problem for the Laplace equation, or even PDE. The computation of the solution of the Dirichlet problem (2.12) is a convenient method (arguably the easiest) to prove the result.

Theorem 2.5.1. *Let f be a 2π-periodic function. Let a_k be its Fourier coefficients, given by the formula*

$$a_k := \frac{1}{2\pi} \int_0^{2\pi} f(\theta) e^{-ik\theta} \, \mathrm{d}\theta.$$

Assume $\sum_{k \in \mathbb{Z}} |a_k| < \infty$. Then the Fourier series

$$\sum_{k=-\infty}^{\infty} a_k e^{ik\theta}$$

converges uniformly to $f(\theta)$.

Proof. Since $\sum_{k \in \mathbb{Z}} |a_k| < \infty$ and $r \in [0,1]$, the series

$$\sum_{k=-\infty}^{\infty} a_k r^{|k|} e^{ik\theta}$$

converges uniformly for $r \in [0,1]$ and $\theta \in [0, 2\pi]$. Since it converges uniformly, the limit must be continuous in $\overline{B_1}$. Let us call this limit $u(r, \theta)$.

We computed above that for $r < 1$,

$$u(r,\theta) = \sum_{k=-\infty}^{\infty} a_k r^{|k|} e^{ik\theta} = \int_0^{2\pi} f(\sigma) P(r, \theta - \sigma) \, \mathrm{d}\sigma.$$

Using the continuity of u and (2.18), we have

$$\sum_{k=-\infty}^{\infty} a_k e^{ik\theta} = u(1,\theta) = \lim_{r \to 1} u(r,\theta) = \lim_{r \to 1} \int_0^{2\pi} f(\sigma) P(r, \theta - \sigma) \, \mathrm{d}\sigma = f(\theta).$$

This concludes the proof. □

Remark 2.5.2. Combining Theorem 2.5.1 with Proposition 2.3.1, we deduce that the Fourier series expansion of any periodic C^2 function converges uniformly to the same function. The result also holds under more general assumptions. For example, it is possible to prove that the function f is the uniform limit of its Fourier series representation whenever f is Hölder continuous. Moreover, by considering other notions of convergence, we can write f as the limit of its Fourier series representation even for very irregular functions.

2.6 Basic properties of Fourier series

Fourier series are a fundamental tool in PDEs and analysis in general. Because of that, it is worth studying some of their properties as a standalone subject.

We start with some of the basic artithmetic properties of Fourier series.

Proposition 2.6.1. *Assume that all the functions involved in this proposition are sufficiently regular so that their Fourier series converge.*

1. *Linearity: If a_k and b_k are the Fourier coefficients of the functions f and g respectively, then for any scalar $\lambda \in \mathbb{C}$, $a_k + \lambda b_k$ are the Fourier coefficients of $f + \lambda g$.*
2. *Product: If a_k and b_k are the Fourier coefficients of the functions f and g respectively, then the Fourier coefficients of the product fg equals c_k where*
$$c_k = \sum_{j \in \mathbb{Z}} a_{k-j} b_j.$$
3. *Convolution: If a_k and b_k are the Fourier coefficients of the functions f and g respectively, then their product $a_k b_k$ are the Fourier coefficients of the convolution $f * g$ defined by*
$$f * g(x) = \frac{1}{2\pi} \int_0^{2\pi} f(x-y) g(y) \, \mathrm{d}y.$$
4. *Derivatives: If a_k are the Fourier coefficients of the function f, then ika_k are the Fourier coefficients of its derivative f'.*

All these properties are relatively simple consequences of the definitions, so we leave them as exercises. The last item relating the Fourier series of a Function with its derivative is what makes them particularly useful for differential equations. Differentiation of a function becomes multiplication by ik in the *Fourier side*. Therefore, a differential equation turns into an algebraic equation when we inspect the Fourier coefficients of its solution.

The Fourier series gives us a correspondence between periodic functions f and sequences $\{a_k\}_{k \in \mathbb{Z}}$. We can think of the values of $\{a_k\}$ as the coordinates of f with respect to the basis e^{ikx}. The next theorem gives us an important identity. It says that, in some sense, the functions e^{ikx} form an orthogonal basis of the space of periodic functions (more precisely in L^2).

Theorem 2.6.2 (Parseval identity)**.** *Let a_k and b_k be the Fourier coefficients of the functions f and g respectively. Then, the following identity holds*
$$\int_0^{2\pi} f(x)\overline{g(x)} \, \mathrm{d}x = 2\pi \sum_{k \in \mathbb{Z}} a_k \overline{b_k}. \tag{2.19}$$

Proof. Writing the Fourier expansion for f and g, we obtain
$$\begin{aligned} \int_0^{2\pi} f(x)\overline{g(x)} \, \mathrm{d}x &= \int_0^{2\pi} \left(\sum_{k \in \mathbb{Z}} a_k e^{ikx} \right) \left(\sum_{m \in \mathbb{Z}} \overline{b_m} e^{-imx} \right) \, \mathrm{d}x, \\ &= \sum_{k \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \left(\int_0^{2\pi} e^{i(k-m)x} \, \mathrm{d}x \right) a_k \overline{b_m}. \end{aligned}$$
The last integral equals to 2π whenever $k = m$ and zero otherwise, thus, only the terms where $k = m$ are nonzero and we obtain the desired result. □

Remark 2.6.3. [for those familiar with the L^2 space] Technically, we have only proved Theorem 2.6.2 for C^2 periodic functions. However, this is a dense subset of $L^2([0, 2\pi])$. Parseval's identity (2.19) extends to all functions in L^2 by continuity. Effectively, Theorem 2.6.2 is saying that the map from a function to its Fourier coefficients gives us an isometric isomorphism from $L^2([0, 2\pi])$ to ℓ^2 (modulo the 2π constant factor).

2.7 Other intervals and boundary conditions

2.7.1 Periods of variable length

The Fourier series representation of Theorem 2.5.1, and the subsequent propositions, applies to periodic functions in the interval $[0, 2\pi]$. If a function f is periodic on an interval $[0, L]$, for any arbitrary length

$L > 0$, one can apply the same technique to the scaled function $\tilde{f}(x) = f(Lx/2\pi)$. Indeed, Theorem 2.5.1 applied to $\tilde{f}$ says that

$$\tilde{f}(x) = \sum_{k=-\infty}^{\infty} a_k e^{ikx},$$

where

$$a_k = \frac{1}{2\pi} \int_0^{2\pi} \tilde{f}(x) e^{-ikx} \, \mathrm{d}x.$$

Changing variables, the two formulas above become

$$f(x) = \sum_{k=-\infty}^{\infty} a_k e^{2\pi ikx/L}, \tag{2.20}$$

where

$$a_k = \frac{1}{L} \int_0^{L} f(x) e^{-2\pi ikx/L} \, \mathrm{d}x.$$

The formulas in Proposition 2.6.1 and Theorem 2.6.2 are also scaled accordingly. Consequently, we can apply Fourier series techniques to periodic functions with any period length.

2.7.2 Functions on an interval that vanish at endpoints

We can use a Fourier series representation of a function f defined in a finite interval (say $[0,1]$) by extending it periodically. This has to be done carefully in order to enforce boundary conditions at the endpoints 0 and 1. Let us start by considering the case of a function $f : [0,1] \to \mathbb{R}$ that vanishes at the endpoints: $f(0) = f(1) = 0$. This was the class of functions that was relevant for our computation of the solution to the heat equation using the method of separation of variables. Assume that f is C^1 in this interval $[0,1]$. We extend f to the interval $[-1,0]$ oddly. That is

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in [0,1], \\ -f(-x) & \text{if } x \in [-1,0]. \end{cases}$$

The condition $f(0) = 0$ ensures the continuity of both $\tilde{f}$ and $\tilde{f}'$ through the point 0. Now we extend the function periodically, with period 2.

Figure 2.1: There should be a picture here :)

The extended function $\tilde{f}$ will be C^1 in the full real line $\mathbb{R}$. It is C^1 at the even integers because of the condition $f(0) = 0$. It is C^1 at the odd integers because of the condition $f(1) = 0$. The second derivative of $\tilde{f}$ will typically have a jump discontinuity at the integer numbers. We can argue it is still a bounded function. We now use the Fourier series representation of $\tilde{f}$ and restrict it to $[0,1]$. We have

$$f(x) = \sum_{k=-\infty}^{\infty} a_k e^{\pi ikx},$$

where

$$a_k = \frac{1}{2} \int_{-1}^{1} \tilde{f}(x) e^{-\pi ikx} \, \mathrm{d}x.$$

Since the extended function $\tilde{f}$ is odd, the coefficients satisfy $a_{-k} = -a_k$ for all $k \in \mathbb{Z}$. Therefore, the formula above becomes

$$f(x) = \sum_{k=1}^{\infty} a_k \left(e^{\pi ikx} - e^{-\pi ikx} \right).$$

Recalling the formula $\sin\theta = (e^{i\theta} - e^{-i\theta})/(2i)$, and setting $\tilde{a}_k = a_k/(2i)$, we see that we have just expressed every function $f : [0,1] \to \mathbb{R}$ that vanishes on 0 and 1 as

$$f(x) = \sum_{k=1}^{\infty} \tilde{a}_k \sin(\pi k x). \tag{2.21}$$

This was the representation that was needed in the computation of the solution of the heat equation by separation of variables.

Remark 2.7.1. One might rightfully wonder why we do not simply extend the function f periodically with period one. If we know $f(0) = f(1) = 0$, this extended function will be continuous but not C^1. Yet, a Fourier series like (2.20) with $L = 1$ does converge to f uniformly. The biggest problem with that idea is the expression (2.20) does not ensure the right boundary values. Thus, if we attempted to solve the heat equation with Dirichlet boundary conditions by evolving the coefficients in (2.20), the Dirichlet boundary condition at $x = 0$ and $x = 1$ would not necessarily hold for $t > 0$.

A similar analysis as above can be done for functions f so that $f'(0) = f'(1) = 0$. In that case, we would make an even reflection to $[-1, 0]$ and then extend periodically with period 2. The final formula turns out to be a series of cosines:

$$f(x) = \sum_{k=0}^{\infty} a_k \cos(\pi k x).$$

We leave the details of this computation up to the reader.

2.8 The Fourier transform

So far, we explored the Fourier series. They allowed us to write an L-periodic function in an interval as an infinite sum of trigonometric functions:

$$f(x) = \sum_{k \in \mathbb{Z}} a_k e^{2\pi i (k/L) x}.$$

Here L is the length of the period, and k ranges among all integers. Note that as $L \to \infty$, the values of k/L form a thinner mesh in $\mathbb{R}$.

The Fourier transform is a version of the same construction but for functions defined in the whole real line which are not assumed to be periodic. It can be understood formally as a limit of the Fourier series as the length of the period L goes to infinity and the possible values of k/L become a continuum of values. We explain this limit process in fair detail below.

Given $f : \mathbb{R} \to \mathbb{C}$, we define its Fourier transform $\hat{f}$ by the formula

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i \xi x} \, \mathrm{d}x.$$

We also define the inverse-Fourier transform $\check{g}$ by

$$\check{g}(x) = \int_{\mathbb{R}} g(\xi) e^{2\pi i \xi x} \, \mathrm{d}\xi.$$

Our main objective will be to prove that $\check{\hat{f}}(x) = f(x)$.

Note that the definitions make sense as long as f and g are integrable functions on $\mathbb{R}$. However, for some proofs it will be convenient to make stronger regularity assumptions.

2.8.1 Obtaining the Fourier transform as a limit of Fourier series with long periods

The definition of the Fourier transform may look artificial at first. Why would we even think of such a formula and why does it make sense to expect $\check{\hat{f}} = f$? In this section we explain why it follows naturally as a limit of Fourier series as the period length L of the function converges to infinity.

This section is only motivational. It can be safely skipped without any logical consequence.

The computation below is a **non rigorous** justification that $\check{\hat{f}} = f$. Everything will be proved rigorously in a later section.

Recall that the coefficients a_k in the Fourier series of an L-periodic function f are computed with the formula

$$a_k = \frac{1}{L} \int_{-L/2}^{L/2} f(x) e^{-2\pi i (k/L) x} \, \mathrm{d}x.$$

And then f is recovered from a_k by the series

$$f(x) = \sum_{k \in \mathbb{Z}} a_k e^{2\pi i (k/L) x}.$$

The same computation can be done with an arbitrary function $f : \mathbb{R} \to \mathbb{C}$. But then the Fourier series will coincide with f only in the interval $(-L/2, L/2)$.

Suppose f is integrable in $\mathbb{R}$. Let $\hat{f} : \mathbb{R} \to \mathbb{C}$ be its Fourier transform.

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i \xi x} \, \mathrm{d}x.$$

Naturally, this function $\hat{f}$ is exactly the following limit

$$\hat{f}(\xi) = \lim_{L \to \infty} L a_{\xi L},$$

where L must be taken to range among those values which make $\xi L \in \mathbb{Z}$. In other words, for large L, we have that

$$a_k \approx \frac{1}{L} \hat{f}(k/L).$$

This is actually an equality if f is supported inside the interval $(-L/2, L/2)$.

Therefore, for large L we should have

$$f(x) \approx \sum_{k \in Z} \frac{1}{L} \hat{f}(k/L) e^{2\pi i (k/L) x}$$

This last expression is nothing but a Riemann sum for the partition of $\mathbb{R}$ given by $\xi_k = k/L$, with $\Delta \xi_k = 1/L$. As $L \to \infty$, the Riemann sum converges to the value of the integral.

$$\lim_{L \to \infty} \sum_{k \in Z} \frac{1}{L} \hat{f}(k/L) e^{2\pi i (k/L) x} = \int_{\mathbb{R}} \hat{f}(\xi) e^{2\pi i \xi x} \, \mathrm{d}x$$

Thus, we finally obtain that by taking $L \to \infty$, we get

$$f(x) = \int_{\mathbb{R}} \hat{f}(\xi) e^{2\pi i \xi x} \, \mathrm{d}\xi.$$

2.8.2 Basic properties of the Fourier transform

Proposition 2.8.1. *The Fourier transform satisfies the following basic properties.*

1. *Linearity:* $(f_1 + \lambda f_2)^\wedge = \hat{f}_1 + \lambda \hat{f}_2$ *and* $(g_1 + \lambda g_2)^\vee = \check{g}_1 + \lambda \check{g}_2$

2. *Translation:* $f(\cdot - h)^\wedge(\xi) = \hat{f}(\xi)e^{-2\pi i h\xi}$ *and* $g(\cdot - h)^\vee(x) = \check{g}(x)e^{2\pi i h x}$
3. *Modulation:* $\left(f(\cdot)e^{2\pi i h\cdot}\right)^\wedge(\xi) = \hat{f}(\xi - h)$ *and* $\left(g(\cdot)e^{-2\pi i h\cdot}\right)^\vee(x) = \check{g}(x - h)$.
4. *Scaling:* $(f(a\cdot))^\wedge(\xi) = \frac{1}{a}\hat{f}\left(\frac{\xi}{a}\right)$ *and* $(g(a\cdot))^\vee(x) = \frac{1}{a}\check{g}\left(\frac{x}{a}\right)$

Here, we only assume that the functions f or g are integrable in $\mathbb{R}$.

Note: where we write $(f(\cdot)e^{2\pi i h\cdot})$ we mean the function $H(x) = f(x)e^{2\pi i h x}$. The same logic applies to other uses of the "dot" notation.

The proof of all items in Proposition 2.8.1 follows from a simple computation using the definition of the Fourier transform and inverse Fourier transform. We do only the last one here, and for the case of the Fourier transform only. All the others are left as an exercise.

We write the formula for $(f(a\cdot))^\wedge$ from the definition

$$(f(a\cdot))^\wedge(\xi) = \int_{\mathbb{R}} f(ax)e^{-2\pi i\xi x}\,\mathrm{d}x,$$

Making the change of variables $y = ax$,

$$\begin{aligned} &= \int_{\mathbb{R}} f(y)e^{-2\pi i\xi\frac{y}{a}}\,\frac{\mathrm{d}y}{a}, \\ &= \frac{1}{a}\int_{\mathbb{R}} f(y)e^{-2\pi i\frac{\xi}{a}y}\,\mathrm{d}y = \frac{1}{a}\hat{f}\left(\frac{\xi}{a}\right). \end{aligned}$$

The following properties are also useful. They will not be used to prove Theorem 2.8.7 (which says that $\check{\hat{f}} = f$), but they are very important for the applications of the Fourier transform.

We start with a semi-obvious statement.

Proposition 2.8.2. *If f is integrable on $\mathbb{R}$, then $\hat{f}$ is a bounded function.*

Proof. We observe that for any $\xi \in \mathbb{R}$,

$$|\hat{f}(\xi)| = \left|\int_{\mathbb{R}} f(x)e^{-2\pi i\xi x}\,\mathrm{d}x\right| \le \int_{\mathbb{R}} |f(x)|\,\mathrm{d}x.$$

□

Proposition 2.8.3. *In the following, always assume that f and g are integrable in $\mathbb{R}$.*

- *If f is differentiable and f' is integrable in $\mathbb{R}$, then $\widehat{f'}(\xi) = 2\pi i\xi\hat{f}(\xi)$.*
- *If $xf(x)$ is integrable in $\mathbb{R}$, then $\hat{f}$ is differentiable and $(\hat{f})'(\xi) = -2\pi i(\cdot f(\cdot))^\wedge(\xi)$.*
- *If g is differentiable and g' is integrable in $\mathbb{R}$, then $\check{g'}(x) = -\,2\pi i x\check{g}(x)$.*
- *If $\xi g(\xi)$ is integrable in $\mathbb{R}$, then $\check{g}$ is differentiable and $(\check{g})'(x) = 2\pi i(\cdot g(\cdot))^\vee(x)$.*

Note that after we prove that $\check{\hat{f}} = f$ one could argue that the first item in the last proposition implies all the others. In any case, the four items are proved using essentially the same computation based on an integration by parts trick. We prove the first one only, the other three are left as an exercise.

Proof. We write the definition of $\widehat{f'}(\xi)$,

$$\begin{aligned} \widehat{f'}(\xi) &= \int_{\mathbb{R}} f'(x)e^{-2\pi i\xi x}\,\mathrm{d}x, \\ &= \lim_{R\to\infty}\int_{-R}^{R} f'(x)e^{-2\pi i\xi x}\,\mathrm{d}x, \end{aligned}$$

We integrate by parts with $du = f'(x)\,\mathrm{d}x$ and $v = e^{-2\pi i\xi x}$,

$$= \lim_{R\to\infty} f(R)e^{-2\pi i\xi R} - f(-R)e^{2\pi i\xi R} + \int_{-R}^{R} f(x)2\pi i\xi e^{-2\pi i\xi x}\,\mathrm{d}x,$$

Note that since f is integrable in $\mathbb{R}$, then necessarily $\lim_{R\to\pm\infty} f(x) = 0$ at least up to subsequences. So we can pass to the limit and get,

$$\begin{aligned} &= 2\pi i\xi \int_{-\infty}^{\infty} f(x)e^{-2\pi i\xi x}\,\mathrm{d}x, \\ &= 2\pi i\xi \hat{f}(\xi). \end{aligned}$$

□

In the previous proposition we can observe an interesting phenomena with the Fourier transform. We see that the regularity of f is transformed into decay of $\hat{f}$ and vice versa. For example, from Proposition 2.8.2, we see that if f is integrable, then $\hat{f}$ is bounded. Combining this fact with Proposition 2.8.3, we see that if f' is integrable then $\hat{f}(\xi) \leq C/|\xi|$ for some constant C. Moreover, if f is k times differentiable and $f^{(k)}$ is integrable in $\mathbb{R}$, then $\hat{f}(\xi) \leq C/|\xi|^k$ for some constant C. Thus, the more derivatives f has, the faster $\hat{f}$ decays at infinity.

2.8.3 Inverse Fourier transform

We now prove that the special function $G(x) = e^{-\pi x^2}$ is the Fourier transform of itself.

Proposition 2.8.4. *The function $G(x) = e^{-\pi x^2}$ satisfies $\hat{G}(\xi) = G(\xi)$. Also $\check{G}(x) = G(x)$.*

Proof. Let us analyze the properties that characterize the Gaussian function G. We prove that G coincides with its Fourier transform by showing that they both satisfy the same differential equation.

One can see that $G(x)$ is characterized as the only function that satisfies the following ordinary differential equation

$$G'(x) = -2\pi x G(x), \tag{2.22}$$

with $G(0) = 1$.

According to Proposition 2.8.3, the Fourier transform of G' is $2\pi i\xi\hat{G}(\xi)$. Moreover, also following Proposition 2.8.3, the Fourier transform of $-2\pi x G(x)$ is $-i\hat{G}'(\xi)$. Taking the Fourier transform of both sides of the equality (2.22), we obtain $-2\pi\xi\hat{G}(\xi) = \hat{G}'(\xi)$. It is the same as the differential equation (2.22) satisfied by G.

It remains to verify that $\hat{G}(0) = 1$. This follows from the well known identity

$$\int_{\mathbb{R}} e^{-\pi x^2}\,\mathrm{d}x = 1.$$

The computation of this integral is one of the most common exercises in vector calculus (see Exercise 2.12).

We have thus verified that $\hat{G}$ and G satisfy the same ODE (2.22) with $G(0) = \hat{G}(0) = 1$. Therefore, they must be the same function. The proof for $\check{G}$ is identical. □

Corollary 2.8.5. *The function $g_{a,x}(\xi) = e^{-2\pi i x\xi}G(a\xi)$ satisfies*

$$\check{g}_{a,x}(y) = \frac{1}{a} G\left(\frac{y-x}{a}\right)$$

Proof. This is a combination of Proposition 2.8.4 with items 3 and 4 in Proposition 2.8.1. □

The following proposition can be understood in a functional analysis context as that the Fourier transform and the inverse Fourier transform are adjoint operators.

Proposition 2.8.6. *Assume both $|f|$ and $|g|$ are integrable, then we have the identity*

$$\int_{\mathbb{R}} \hat{f}(\xi)\overline{g(\xi)} \,\mathrm{d}\xi = \int_{\mathbb{R}} f(x)\overline{\check{g}(x)} \,\mathrm{d}x.$$

Proof. For the proof it suffices to write the definitions of the Fourier and inverse Fourier transforms and change the order of integration

$$\begin{aligned}
\int_{\mathbb{R}} \hat{f}(\xi)\overline{g(\xi)} \,\mathrm{d}\xi &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x) e^{-2\pi i \xi x} \,\mathrm{d}x \right) \overline{g(\xi)} \,\mathrm{d}\xi \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \overline{g(\xi)} e^{-2\pi i \xi x} \,\mathrm{d}\xi \right) f(x) \,\mathrm{d}x \\
&= \int_{\mathbb{R}} \overline{\check{g}(x)} f(x) \,\mathrm{d}x
\end{aligned}$$

Note: the change in the order of integration is justified by Fubini's theorem since

$$\iint_{\mathbb{R}\times\mathbb{R}} \left| f(x) e^{-2\pi i \xi x} \overline{g(\xi)} \right| \,\mathrm{d}x \,\mathrm{d}\xi = \left(\int_{\mathbb{R}} |f(x)| \,\mathrm{d}x \right) \left(\int_{\mathbb{R}} |g(\xi)| \,\mathrm{d}\xi \right) < +\infty.$$

□

The following result is the fact that $\check{\hat{f}} = f$. For now, we will assume that both f and $\hat{f}$ are continuous and integrable. We will see later that if $f \in \mathcal{S}$, then both of these conditions are satisfied.

Theorem 2.8.7. *Assume both f and $\hat{f}$ are continuous and integrable. Then $\check{\hat{f}}(x) = f(x)$.*

Proof. We use the function g from Corollary 2.8.5 in Proposition 2.8.6. We have

$$\int_{\mathbb{R}} \hat{f}(\xi)\overline{g_{a,x}(\xi)} \,\mathrm{d}\xi = \int_{\mathbb{R}} f(y)\overline{\check{g}_{a,x}(y)} \,\mathrm{d}y. \tag{2.23}$$

Note that $\check{g}_{a,x}$ is a Gaussian centered at x and scaled by the factor a in a way that $\int_{\mathbb{R}} \check{g}_{a,x} = 1$ for all a. This is the same as the Heat kernel for $t = \sqrt{a}$. Recall (for example from our study of the heat equation) that we have

$$\int_{\mathbb{R}} f(y)\overline{\check{g}_{a,x}(y)} \,\mathrm{d}y = \int_{\mathbb{R}} f(y) \frac{1}{a} G\left(\frac{y-x}{a}\right) \,\mathrm{d}y \to f(x) \quad \text{as } a \to 0.$$

Thus, we can recover $f(x)$ taking $a \to 0$ in (2.23). We then obtain

$$f(x) = \lim_{a\to 0} \int_{\mathbb{R}} \hat{f}(\xi)\overline{g_{a,x}(\xi)} \,\mathrm{d}\xi = \lim_{a\to 0} \int_{\mathbb{R}} \hat{f}(\xi)\overline{e^{-2\pi i x\xi} G(a\xi)} \,\mathrm{d}\xi = \int_{\mathbb{R}} \hat{f}(\xi) e^{2\pi i x \xi} \,\mathrm{d}\xi.$$

This finishes the proof of our theorem. □

Note that the last identity is justified by the dominated convergence theorem given that we assume that $|\hat{f}(\xi)|$ is integrable.

Note that from Propositions 2.8.2 and 2.8.3, if f is integrable and twice differentiable with f'' integrable, then $\hat{f}$ is also integrable and Theorem 2.8.7 applies.

2.8.4 Parseval's identity

The following result is known as Parseval's identity.

Proposition 2.8.8 (Parseval's identity)**.** *Assume f and $\hat{f}$ are both continuous and integrable. Then*

$$\int_{\mathbb{R}} |f|^2 \,\mathrm{d}x = \int_{\mathbb{R}} |\hat{f}|^2 \,\mathrm{d}y.$$

Proof. This results follows immediately from Proposition 2.8.6 using $g(\xi) = \hat{f}(\xi)$ and Theorem 2.8.7. □

Remark 2.8.9. Parseval's identity allows us to extend the definition of the Fourier transform to the L^2 space (that is the space of all functions f so that $|f|^2$ is integrable). In order to do that, we approximate f by a sequence of smooth, compactly supported, functions f_n so that $f_n \to f$ in L^2. By Parseval's identity, we have

$$\|\hat{f}_n - \hat{f}_m\|_{L^2} = \|f_n - f_m\|_{L^2}.$$

Therefore, $\hat{f}_n$ is a Cauchy sequence in L^2 and we can define $\hat{f}$ as its limit.

It is not hard to see that this definition of $\hat{f}$ does not depend on the choice of the sequence f_n and it inherits all the relevant properties of the Fourier transform. In particular, the Fourier transform is a isometric isomorphism in $L^2(\mathbb{R})$.

Multiplications and convolutions

Let f and g be two bounded integrable functions. We define the convolution of f and g as

$$f \star g(x) = \int_{\mathbb{R}} f(y)g(x-y)\,\mathrm{d}y = \int_{\mathbb{R}} f(x-y)g(y)\,\mathrm{d}y.$$

An important property of the Fourier transform is that it exchanges the operations of multiplication and convolution.

Proposition 2.8.10. *Let f and g be two bounded integrable functions. Then*

$$(f \star g)^\wedge(\xi) = \hat{f}(\xi)\,\hat{g}(\xi).$$

Proof. We write the expression of $(f \star g)^\wedge$ as a double integral and apply Fubini's theorem.

$$\begin{aligned}
(f \star g)^\wedge(\xi) &= \int\int f(y)g(x-y)e^{-2\pi i x\xi}\,\mathrm{d}y\,\mathrm{d}x, \\
&= \int\int \left(f(y)e^{-2\pi i y\xi}\right)\left(g(x-y)e^{-2\pi i(x-y)\xi}\right)\,\mathrm{d}y\,\mathrm{d}x, \\
&= \int\int \left(f(y)e^{-2\pi i y\xi}\right)\left(g(x-y)e^{-2\pi i(x-y)\xi}\right)\,\mathrm{d}x\,\mathrm{d}y, \\
&= \hat{g}(\xi)\int f(y)e^{-2\pi i y\xi}\,\mathrm{d}y, \\
&= \hat{g}(\xi)\hat{f}(\xi).
\end{aligned}$$

□

2.8.5 Applications of the Fourier transform to PDE

The Fourier transform tends to simplify the study of linear PDEs with constant coefficients in the full real line. We show a few examples below.

The heat kernel

Consider the heat equation in the full real line.

$$\begin{aligned}
u_t &= u_{xx}, \\
u(0,x) &= u_0(x).
\end{aligned}$$

For each fixed value of t, we take the Fourier transform of $u(t,x)$ as a function of x. We call this $\hat{u}(t,\xi)$. Using Proposition 2.8.3, we deduce

$$\begin{aligned}
\hat{u}_t(t,\xi) &= -4\pi^2|\xi|^2\hat{u}(t,\xi), \\
\hat{u}(0,\xi) &= \hat{u}_0(\xi).
\end{aligned}$$

For each value of ξ, this defines an ODE in t whose solution is

$$\hat{u}(t,\xi) = e^{-4\pi^2|\xi|^2 t}\hat{u}_0(\xi) \tag{2.24}$$

Therefore, the value of $u(x,t)$ must be given by the convolution

$$u(t,x) = H \star u_0(x),$$

where

$$\hat{H}(\xi) = e^{-4\pi^2|\xi|^2 t}.$$

From Propositions 2.8.4 and 2.8.1, H is

$$H(x) = \frac{1}{(4\pi t)^{1/2}} e^{-|x|^2/(4t)}. \tag{2.25}$$

The wave equation

Let us now focus on the wave equation in the real line.

$$\begin{aligned} u_{tt} &= u_{xx}, \\ u(0,x) &= f(x), \\ u_t(0,x) &= g(x). \end{aligned}$$

Again, we write $\hat{u}$ to denote the Fourier transform of u respect to x. We obtain.

$$\begin{aligned} \hat{u}_{tt}(t,\xi) &= -4\pi^2|\xi|^2\hat{u}(t,\xi), \\ \hat{u}(0,\xi) &= \hat{f}(\xi), \\ \hat{u}_t(0,\xi) &= \hat{g}(\xi). \end{aligned}$$

For each fixed value of ξ, the equation gives us an ODE with respect to t. Solving this ODE, we obtain

$$\hat{u}(t,\xi) = \cos(2\pi\xi t)\hat{f}(\xi) + \frac{1}{2\pi\xi}\sin(2\pi\xi t)\hat{g}(\xi). \tag{2.26}$$

This computation gives us an expression for the Fourier transform of the solution $\hat{u}$. We can then take the inverse Fourier transform to recover the actual solution u in physical variables. In Exercise 2.15, we use this computation to derive D'Alembert formula for the solution of the wave equation in 1D.

The Schrödinger equation

The Schrödinger equation plays a central role in quantum mechanics. It is an evolution equation for a complex valued function u given by

$$\begin{aligned} iu_t &= u_{xx}, \\ u(0,x) &= u_0(x). \end{aligned}$$

After taking Fourier transform in x, we obtain

$$\begin{aligned} \hat{u}_t(t,\xi) &= i4\pi^2|\xi|^2\hat{u}(t,\xi), \\ \hat{u}(0,\xi) &= \hat{u}_0(\xi). \end{aligned}$$

For each value of ξ, we solve an ODE to obtain

$$\hat{u}(t,\xi) = e^{i4\pi^2|\xi|^2 t}\hat{u}_0(\xi) \tag{2.27}$$

It is the same formula as in the solution of the heat equation but evaluated in it instead of t. The behavior of the equation is very different though. The reason why the solutions to the heat equation are C^∞ is because the right hand side in (2.24) has a very strong decay as $\xi \to \infty$. The right hand side in (2.27) has the same modulus as $\hat{u}_0(\xi)$ for any value of ξ and $t \in \mathbb{R}$. Therefore, a solution to the Schrödinger equation is as regular as its initial data, not more.

2.8.6 The Fourier transform of functions in higher dimension.

The definitions and results above can be extended to higher dimensions using the following definitions.

Given $f : \mathbb{R}^n \to \mathbb{C}$, we define its Fourier transform $\hat{f}$ by the formula

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi \cdot x} \, \mathrm{d}x.$$

We also define the inverse-Fourier transform $\check{g}$ by

$$\check{g}(x) = \int_{\mathbb{R}^n} g(\xi) e^{2\pi i \xi \cdot x} \, \mathrm{d}\xi.$$

With these definitions, it is also true that $\check{\hat{f}} = f$. The natural modified versions of Propositions 2.8.1, 2.8.3, 2.8.2, 2.8.4 and 2.8.10 also apply.

2.9 Exercises

Exercise 2.1. *Prove the following properties of Fourier series. Throughout this question, the function f and g are 2π periodic, bounded and integrable.*

1. *Linearity: If a_k and b_k are the Fourier coefficients of the functions f and g respectively, then for any scalar $\lambda \in \mathbb{C}$, $a_k + \lambda b_k$ are the Fourier coefficients of $f + \lambda g$.*
2. *Product: If a_k and b_k are the Fourier coefficients of the functions f and g respectively, then the Fourier coefficients of the product fg equals c_k where*
$$c_k = \sum_{i \in \mathbb{Z}} a_{k-i} b_i.$$
3. *Convolution: If a_k and b_k are the Fourier coefficients of the functions f and g respectively, then their product $a_k b_k$ are the Fourier coefficients of the convolution $f * g$ defined by*
$$f * g(x) = \frac{1}{2\pi} \int_0^{2\pi} f(x-y) g(y) \, \mathrm{d}y.$$
4. *Derivatives: If $f \in C^1$ and a_k are the Fourier coefficients of the function f, then ika_k are the Fourier coefficients of its derivative f'.*
5. *Derivatives again: If f is bounded and $|ka_k|$ is summable. Then f is actually C^1 and ika_k are the Fourier coefficients of f'.*

Exercise 2.2. *Write down explicitly a solution to the following equation*

$$\begin{aligned} u_t - u_{xx} + u &= 0, && \text{for } t > 0, x \in (0,1), \\ u(t,0) = u(t,1) &= 0, && \text{for } t > 0, \\ u(0,x) &= \sin(\pi x) + \sin(3\pi x). \end{aligned}$$

Exercise 2.3. *Let us consider the Wave equation with Dirichlet boundary conditions. We look for a function $u : [0,+\infty) \times [0,1] \to \mathbb{R}$ which solves*

$$\begin{aligned} u_{tt} &= u_{xx}, \\ u(t,0) = u(t,1) &= 0, \\ u(0,x) &= f(x), \\ u_t(0,x) &= g(x). \end{aligned}$$

Assume that the functions f and g are written as the following trigonometric series.

$$f(x) = \sum_{k=1}^{\infty} a_k \sin(\pi k x), \qquad g(x) = \sum_{k=1}^{\infty} b_k \sin(\pi k x).$$

Let us represent u as a time dependent trigonometric series.

$$u(t,x) = \sum_{k=1}^{\infty} c_k(t) \sin(\pi k x).$$

Write a formula for $c_k(t)$ in terms of a_k, b_k and t.

Exercise 2.4. *Let $u : [0,\infty) \times [0,1] \to \mathbb{R}$ be a smooth function with $u(t,0) = u(t,1) = 0$ for all $t \geq 0$. Let us write u in terms of its Fourier series for every value of t. That is*

$$u(t,x) = \sum_{k=1}^{\infty} a_k(t) \sin(\pi k x),$$

where

$$a_k(t) = 2 \int_0^1 \sin(\pi k x) u(t,x) \,\mathrm{d}x.$$

Verify that the heat equation $u_t - u_{xx} = 0$ is equivalent to the following family of ODEs:

$$a_k'(t) = -\pi^2 k^2 a_k(t).$$

Use this to justify the uniqueness of solutions to the heat equation for any initial data $u(0,x) = f(x)$.

Exercise 2.5. *Let $u(t,x)$ be a solution of the heat equation in the unit interval with Dirichlet boundary conditions.*

$$u_t = u_{xx},$$
$$u(t,0) = u(t,1) = 0.$$

Use the Fourier series representatin of u to prove that

$$H(t) := \int_0^1 u(t,x)^2 \,\mathrm{d}x,$$

is non increasing with respect to t.

Exercise 2.6. *In our first example of the method of separation of variables for the heat equation, we obtained that there was a function $H(t,x,y)$ such that a solution to*

$$\begin{aligned} &u_t - u_{xx} = 0 && \text{for } x \in (0,1), t > 0,\\ &u(t,0) = u(t,1) = 0 && \text{for } t > 0,\\ &u(0,x) = f(x) && \text{for } x \in (0,1), \end{aligned}$$

was given by the formula

$$u(t,x) = \int_0^1 f(y) H(t,x,y) \,\mathrm{d}y.$$

Prove that a solution to the equation

$$\begin{aligned} &u_t - u_{xx} = g(t,x) && \text{for } x \in (0,1), t > 0,\\ &u(t,0) = u(t,1) = 0 && \text{for } t > 0,\\ &u(0,x) = f(x) && \text{for } x \in (0,1), \end{aligned}$$

is given by the formula

$$u(t,x) = \int_0^1 f(y)H(t,x,y)\, \mathrm{d}y + \int_0^t \int_0^1 g(s,y)H(t-s,x,y)\, \mathrm{d}y\, \mathrm{d}s.$$

Hint. *If you took ODEs following the book of Hirsch, Smale and Devaney, this should remind you of the formula in the Theorem of page 131 (Variation of parameters).*

Exercise 2.7. *Let f be a continuous function with Fourier coefficients a_k. Assume that $\sum |k|^m |a_k| < +\infty$. Prove that f is C^m.*

Exercise 2.8. *Let $f : [0,1] \to \mathbb{R}$ be a function whose Fourier coefficients converge absolutely (i.e. $\sum |a_k| < +\infty$). Prove that the solution u to the heat equation*

$$\begin{aligned} &u_t - u_{xx} = 0 && \text{for } x \in (0,1), t > 0,\\ &u(t,0) = u(t,1) = 0 && \text{for } t > 0,\\ &u(0,x) = f(x) && \text{for } x \in (0,1), \end{aligned}$$

is C^∞ in $(0,\infty) \times (0,1)$.

Exercise 2.9. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a periodic function. That means that $f(x+a, y+b) = f(x,y)$ for any two integers a and b. Assume that $f \in C^2(\mathbb{R}^2)$. Prove that*

$$f(x,y) = \sum_{m,n \in \mathbb{Z}^2} a_{mn} e^{2\pi i (mx+ny)},$$

where

$$a_{mn} = \int_{[0,1]^2} f(x,y) e^{-2\pi i (mx+ny)}\, \mathrm{d}x\, \mathrm{d}y.$$

Hint. Use the corresponding one dimensional result as a fact. Use that for every value of $y \in [0,1]$, the function $f(\cdot, y)$ is equal to its Fourier series.

Exercise 2.10. *Using separation of variables, write a formula for the solution of the heat equation in a square.*

$$\begin{aligned} u_t &= \Delta u && \text{for } t > 0, (x,y) \in (0,1)^2,\\ u(t,x,y) &= 0 && \text{for } t > 0, (x,y) \in \partial(0,1)^2,\\ u(0,x,y) &= u_0(x,y). \end{aligned}$$

In particular, what would the explicit solution be if $u_0(x,y) = \sin(\pi x)\sin(5\pi y)$?

Exercise 2.11. *For $\lambda \in \mathbb{R}$, consider the equation*

$$\begin{aligned} &u_t = u_{xx}/3 + \lambda u && \text{for } t > 0, x \in (0,1),\\ &u(t,0) = u(t,1) = 0 && \text{for } t > 0,\\ &u(0,x) = x(1-x) && \text{for } x \in (0,1). \end{aligned}$$

For what values of λ does $\lim_{t\to\infty} u(t,1/2) = +\infty$?

Exercise 2.12. *The objective of this exercise is to compute the integral of the Gaussian, which is required in the proof of Proposition 2.8.4. Prove that for any dimension d, the following identity holds.*

$$\int_{\mathbb{R}^d} e^{-\pi |x|^2}\, \mathrm{d}x = 1.$$

Hint. *Note that since* $e^{-\pi|x|^2} = \prod_{i=1}^d e^{-\pi x_i^2}$*, then*

$$\int_{\mathbb{R}^d} e^{-\pi|x|^2}\,\mathrm{d}x = \left(\int_{\mathbb{R}} e^{-\pi x^2}\,\mathrm{d}x\right)^d.$$

Thus, it is enough to do the computation for one particular dimension to conclude the identity for any arbitrary dimension d. Somewhat surprisingly, the computation can be carried out directly in two dimensions more easily than in one dimension. When d = 2, we can compute the double integral using polar coordinates.

Exercise 2.13. *For some given constant* $c > 0$*, consider the following domain which is shaped like a pie slice*

$$D_R = \{(x,y) : x^2 + y^2 < R^2, x > 0, y > 0 \text{ and } y < cx\}.$$

Note that in polar coordinates (r,θ) *it corresponds to* $r \in (0,R)$ *and* $\theta \in (0,\theta_0)$ *where* $\tan\theta_0 = c$*.*

(a) Write a general expression (as a trigonometric series in polar coordinates) for the solution of the Laplace equation with Dirichlet boundary conditions in D_R*.*

$$\begin{aligned} \Delta u &= 0 && \text{in } D_R, \\ u &= 0 && \text{whenever } y = 0 \text{ or } y = cx, \\ u &= f && \text{given in } x^2 + y^2 = R^2. \end{aligned}$$

(b) Assuming the function u from part (a) has finitely many terms in the expression obtained, prove that the function u can be extended as a solution in the infinite slice.

$$\begin{aligned} \Delta u &= 0 && \text{in } D_\infty, \\ u &= 0 && \text{whenever } y = 0 \text{ or } y = cx. \end{aligned}$$

(c) Prove that if the function u from part (b) is nonnegative then only the first term may be nonzero.

(d∗) Let u_1 *and* u_2 *be two nonnegative solutions of*

$$\begin{aligned} \Delta u_i &= 0 && \text{in } D_\infty, \\ u_i &= 0 && \text{whenever } x = 0 \text{ or } y = cx, \end{aligned}$$

for $i = 1, 2$*. Prove that either* $u_2 \equiv 0$ *or there exists* $\lambda > 0$ *so that* $u_1 = \lambda u_2$*.*

Exercise 2.14. *Let* $u : B_1 \to \mathbb{R}$ *be a harmonic function, where* B_1 *is the unit disk in* $\mathbb{R}^2$*. Let* $k \in \mathbb{N}$ *be a number so that the limit*

$$\lim_{r\to 0} r^{-k} \int_{\partial B_r} |u|^2\,\mathrm{d}S$$

is a finite positive real number.

(a) Find all the possible values for k, for all harmonic functions u.

(b) Prove that the quantity

$$M(r) := r^{-k} \int_{\partial B_r} |u|^2\,\mathrm{d}S$$

is monotone increasing.

Hint. Write $u(r,\theta)$ as the series in (2.15). Then, write an expression for $M(r)$ in terms of r and the coefficients $\{a_k\}$.

Exercise 2.15. *The objective of this exercise is to derive Dalembert's formula for the solution of the wave equation in 1D. An alternative derivation, purely in physical variables, will be done in Chapter 5.*

Using Proposition 2.8.1, verify that for any $f : \mathbb{R} \to \mathbb{R}$ *integrable, so that* $\hat{f}$ *is integrable, we have*

- $\cos(2\pi\xi t)\hat{f}(\xi)$ *is the Fourier transform of* $\frac{1}{2}\left(f(x+t)+f(x-t)\right)$.
- $\frac{1}{2\pi\xi}\sin(2\pi\xi t)\hat{g}(\xi)$ *is the Fourier transform of*

$$\frac{1}{2}\int_{-t}^{t} g(x+s)\,\mathrm{d}s.$$

.

Using 2.26, derive D'Alembert formula for the wave equation in 1D that says that the solution to

$$\begin{aligned} u_{tt}(t,x) &= u_{xx}(t,x) \text{ for } t>0, x\in\mathbb{R},\\ u(0,x) &= f(x),\\ u_t(0,x) &= g(x), \end{aligned}$$

is given by

$$u(t,x)=\frac{1}{2}\left(f(x+t)+f(x-t)\right)+\frac{1}{2}\int_{-t}^{t} g(x+s)\,\mathrm{d}s.$$

Exercise 2.16. *We want to solve the heat equation on the half line $x\in(0,\infty)$ with Dirichlet boundary conditions:*

$$\begin{aligned} u_t(t,x) &= u_{xx}(t,x) \text{ for } t>0, x\in(0,\infty),\\ u(0,x) &= f(x),\\ u(t,0) &= 0. \end{aligned}$$

Derive the formula

$$u(t,x)=\int_0^{\infty} f(y)(H(t,x-y)-H(t,x+y))\,\mathrm{d}y,$$

where H is the heat kernel as in (2.25)

Chapter 3

The heat equation

In the first chapter, we discussed an informal derivation of the heat equation as the evolution of temperature at every point in a homogeneous body. The same (or similar) equation is in fact used for a variety of models in physics, chemistry, population dynamics and economics. It is the main equation that represents the notion of diffusion.

The heat equation has a peculiar regularizing effect. It turns out that even for rough initial data, the solution becomes immediately smooth. Moreover, as time goes to infinity, the solution converges to equilibrium. In this chapter we explore some formulas, a priori estimates, and theorems that expose how special the heat equation is.

There are two alternative methods that lead to the uniqueness of solutions: energy methods and the maximum principle. We will start by discussing these methods and their implications. Later, we will compute some formulas and derive a priori estimates.

3.1 Energy methods

Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with piecewise C^1 boundary. We study solutions to the heat equation

$$u_t - \Delta u = 0, \qquad \text{in } (0,\infty) \times \Omega \tag{3.1}$$

The initial value $u(0,x) = f(x)$ is arbitrary (and usually given). We should also provide a boundary condition on $\partial\Omega$. The *Dirichlet* boundary condition is

$$u(t,x) = 0 \qquad \text{when } x \in \partial\Omega. \tag{3.2}$$

Conversely, the *Neumann* boundary condition (alt. the *no flux* boundary condition) is

$$u_\nu(t,x) = 0 \qquad \text{when } x \in \partial\Omega, \tag{3.3}$$

where ν is the unit normal vector to $\partial\Omega$.

There are several quantities associated with the solution of the heat equation that are non increasing in time. The following proposition shows one of the simplest ones.

Proposition 3.1.1. *Let* $u : [0,\infty) \times \Omega \to \mathbb{R}$ *solve the heat equation* (3.1) *with either Dirichlet* (3.2) *or Neumann* (3.3) *boundary conditions. The following quantity*

$$E(t) := \int_\Omega u(t,x)^2 \, \mathrm{d}x,$$

is non increasing in time.

Proof. Let us compute $E'(t)$ by differentiating inside the integral and using the equation.

$$\begin{aligned} E'(t) &= 2\int_\Omega u(t,x)u_t(t,x)\,\mathrm{d}x, \\ &= 2\int_\Omega u(t,x)\Delta u(t,x)\,\mathrm{d}x, \end{aligned}$$

integrating by parts

$$= 2\int_\Omega -|\nabla u(t,x)|^2\,\mathrm{d}x + 2\int_{\partial\Omega} u u_\nu\,\mathrm{d}S.$$

The boundary term vanishes since we assume either $u = 0$ or $u_\nu = 0$ on $\partial\Omega$. The first term is less or equal than zero. Thus, we deduced that $E'(t) \leq 0$. This guarantees that $E(t)$ is non increasing in time. ☐

Corollary 3.1.2 (Uniqueness of solutions)**.** *For any smooth function $f : \Omega \to \mathbb{R}$, the initial value problem*

$$\begin{aligned} u(0,x) &= f(x), \\ u_t - \Delta u &= 0, \qquad \text{in } (0,\infty)\times\Omega, \\ u(t,x) &= 0 \qquad (\text{or } u_\nu = 0) \text{ when } x \in \partial\Omega \end{aligned}$$

has at most one solution u.

Proof. Given two solutions u_1 and u_2, we write $w = u_1 - u_2$. The function w is a solution to the heat equation (3.1) with either Dirichlet (3.2) or Neumann (3.3) boundary condition. We apply Proposition 3.1.1 to w. In this case $E(0) = 0$ since $w(0,x) = 0$. Since E is nonincreasing and nonnegative, we must have $E \equiv 0$. So $w \equiv 0$ and $u_1 = u_2$. ☐

3.2 The maximum principle

In this section we give a version of the maximum principle for the heat equation. We do it in a bounded domain first, and in the full space later.

3.2.1 The maximum principle in a bounded domain

Let $\Omega \subset \mathbb{R}^d$ be a bounded open set. We write Ω_T and $\partial_p\Omega_T$ to denote the parabolic domain and its parabolic boundary respectively.

$$\begin{aligned} \Omega_T &:= (0,T]\times\Omega, \\ \partial_p\Omega_T &= \{(t,x) \in [0,T]\times\overline{\Omega} : t = 0 \text{ or } x \in \partial\Omega\}. \end{aligned}$$

Note that the parabolic boundary $\partial_p\Omega_T$ includes the initial time $t = 0$ but not the final time $t = T$.

The maximum principle (stated precisely as the next theorem) says that any solution of the heat equation achieves its maximum at some point on the parabolic boundary.

Theorem 3.2.1 (Maximum principle)**.** *Let $u : \overline{\Omega_T} \to \mathbb{R}$ be continuous in $\overline{\Omega}_T$ and satisfy*

$$u_t - \Delta u \leq 0 \text{ in } \Omega_T.$$

Then

$$\max_{\overline{\Omega_T}} u = \max_{\partial_p\Omega_T} u.$$

Proof. We do the proof by contradiction. Assume that there exists a point $(t_0, x_0) \in \Omega_T$ so that $u(t_0,x_0) > \max_{\partial_p\Omega_T} u$.

Let $\varepsilon > 0$ be sufficiently small so that

$$u(t_0, x_0) > \max_{\partial_p \Omega_T} u + \varepsilon T. \tag{3.4}$$

We define the function

$$v(t, x) = u(t, x) - \varepsilon t.$$

From (3.4), we deduce that $v(t_0, x_0) \geq v(t, x)$ for any $(t, x) \in \partial_p \Omega_T$. Let $(t_1, x_1) \in \overline{\Omega_T}$ be the point so that

$$v(t_1, x_1) = \max_{\overline{\Omega_T}} v.$$

We know such a point (t_1, x_1) exists because of the compactness of $\overline{\Omega_T}$. Moreover, this point cannot belong to $\partial_p \Omega_T$ because the values of v of the parabolic boundary are all smaller than $v(t_0, x_0)$.

Since v achieves its maximum at this point and $x_1 \in \Omega$, we deduce that $\Delta v(t_1, x_1) = \Delta u(t_1, x_1) \leq 0$.

Likewise, if $t_1 \in (0, T)$ and v achieves its maximum at (t_1, x_1), then $v_t(t_1, x_1) = 0$. It may happen that $t_1 = T$, in which case we can only deduce the inequality $v_t(t_1, x_1) \geq 0$.

Combining the two inequalities above,

$$v_t(t_1, x_1) - \Delta v(t_1, x_1) \geq 0.$$

However, this contradicts the equation for u, since

$$v_t - \Delta v = u_t - \Delta u - \varepsilon \leq -\varepsilon < 0.$$

We arrived to a contradition from the assumption that u achieves an interior value that is strictly larger than all its values on the parabolic boundary. We conclude that the maximum of u must be achieved on the parabolic boundary. □

The maximum principle can be applied to the solution u, and also to $-u$. It immediately implies the *minimum principle*: the minimum of the solution is also achieved on the parabolic boundary. Consequently, if the initial and the boundary values for the heat equation are zero, the solution must vanish everywhere, because its maximum and minimum have to be both zero.

Applying the maximum principle to the difference between two solutions, we derive the following result, that we call *comparison principle*.

Corollary 3.2.2 (Comparison principle)**.** *Let $u, v : \overline{\Omega_T} \to \mathbb{R}$ be two continuous functions in $\overline{\Omega}_T$ so that*

$$u_t - \Delta u \leq v_t - \Delta v \text{ in } \Omega_T,$$
$$u \leq v \text{ on } \partial_p \Omega_T.$$

Then $u \leq v$ in Ω_T.

Proof. Apply Theorem 3.2.1 (the maximum principle) to $u - v$. □

A direct consequence of the comparison principle is the uniqueness of the solutions of the heat equation. We have achieved two independent proofs of uniqueness, one using energy methods, and another using the maximum principle.

3.2.2 The maximum principle in the full space

Now, we want to extend the maximum principle for solutions of the heat equation in the full space. In this case $\Omega = \mathbb{R}^d$. There is no boundary condition but only an initial value at time zero.

The proof of the maximum principle we gave before (for Ω bounded) was based on evaluating the equation at one point where the auxiliary function v achieves a maximum. There is a technical difficulty when Ω is not compact, because the maximum of a bounded continuous function might not be achieved anywhere.

Theorem 3.2.3. *Let $u : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ be a bounded continuous function which satisfies*

$$u_t - \Delta u \le 0 \text{ for all } t \in (0, T].$$

Then

$$\sup_{[0,T]\times\mathbb{R}^d} u = \sup_{x\in\mathbb{R}^d} u(0, x).$$

Proof. Let $\varepsilon > 0$ be an arbitrarily small quantity. Let

$$v(t, x) = \varepsilon \left(t + \frac{|x|^2}{2d} \right) + \left\{ \sup_{y\in\mathbb{R}^d} u(0, y) \right\}.$$

The function v is chosen so that $v_t - \Delta v = 0$. Moreover, $v(t, x)$ grows quadratically as $|x| \to \infty$. Thus, for a large radius R (depending on ε and $\sup |u|$), we have $v \ge u$ outside B_R.

Since we have $v \ge 0$ on $t = 0$ and on $x \in \partial B_R$, we use the comparison principle as in Corollary 3.2.2 with $U = B_R$ to obtain $v \ge u$ everywhere.

$$u(t, x) \le v(t, x) = \varepsilon \left(t + \frac{|x|^2}{2d} \right) + \left\{ \sup_{y\in\mathbb{R}^d} u(0, y) \right\}. \tag{3.5}$$

Taking $\varepsilon \to 0$ in (3.5), we obtain

$$u(t, x) \le \sup_{y\in\mathbb{R}^d} u(0, y).$$

□

As a corollary, we derive the uniqueness of solutions of the initial value problem in the full space.

Corollary 3.2.4. *For any continuous bounded functions $f : \mathbb{R}^d \to \mathbb{R}$ and $g : [0, T] \times \mathbb{R}^d \to \mathbb{R}$, there exists at most one bounded continuous function $u : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ that solves*

$$\begin{aligned} u_t - \Delta u &= g, \quad \text{in } (0, T] \times \mathbb{R}^d, \\ u(0, x) &= f \quad \text{on } \mathbb{R}^d. \end{aligned}$$

Proof. If there were two solutions u_1 and u_2, we get that their difference $w = u_1 - u_2$ is also bounded and solves

$$\begin{aligned} w_t - \Delta w &= 0, \quad \text{in } (0, T] \times \mathbb{R}^d, \\ w(0, x) &= 0 \quad \text{on } \mathbb{R}^d. \end{aligned}$$

The maximum principle tells us that w must be identically zero. □

Note that we established the uniqueness for the heat equation in Corollary 3.2.4 only within the class of **bounded** solutions. The result would not hold without that assumption. There are examples of unbounded solutions to the heat equation in the full space that start from zero and later become nonzero. These pathological solutions have oscillations that grow very rapidly as $x \to \infty$ (See [3, Chapter 7] for the construction).

3.3 Heat kernel

There is a special solution of the heat equation that we will use to compute a formula for the solutions of the equation in the full space. It is called the *fundamental solution* or *heat kernel*.

Loosely speaking, the heat kernel is a solution of the heat equation whose initial data consists of a unit mass concentrated at one point. This solution is invariant by some of the symmetries of the heat equation. We use these symmetries, to explicitly compute the heat kernel below.

3.3.1 Symmetries of the equation

Assume we have a solution u of the heat equation in the full space.

$$u_t = \Delta u \qquad \text{in } (0,\infty)\times\mathbb{R}^d.$$

There are some simple transformations that give us another solution. They are the following

Translation $u(t, x-y)$ is also a solution for any $y \in \mathbb{R}^d$.

Dilation $cu(t,x)$ is also a solution for any value of $c \in \mathbb{R}$.

Scaling $u(b^2 t, bx)$ is also a solution for any value of $b \in \mathbb{R}$.

Rotation $u(t, Qx)$ is also a solution for any rotation matrix (i.e. $Q^t Q = I$).

3.3.2 The fundamental solution

The fundamental solution to the heat equation, which we write $H(t,x)$ is a function which satisfies the following properties.

1. It solves the equation: $H_t(t,x) = \Delta H(t,x)$ for $x \in \mathbb{R}^d$ and $t > 0$.

2. It is non negative and its integral is one.

$$\int_{\mathbb{R}^d} H(t,x)\,\mathrm{d}x = 1 \qquad \text{for every } t > 0. \tag{3.6}$$

   Note that the integral of any solution of the heat equation is constant in time.

3. It is rotationally invariant: $H(t,Qx) = H(t,x)$ for any rotation matrix Q. Therefore, $H(t,x)$ depends only on t and $|x|$.

4. It has the following scale invariance, for all $b > 0$,

$$b^d H(b^2 t, bx) = H(t,x). \tag{3.7}$$

   Note that the first factor b^d cannot have any other power since this property must be compatible with the fact that the integral of $H(t,x)$ with respect to x is constant in t. Indeed, if we make the change of variables $y = rx$, and use the Jacobian formula $\mathrm{d}y = r^d\,\mathrm{d}x$, we get

$$\int_{\mathbb{R}^d} H(t,x)\,\mathrm{d}x = \int_{\mathbb{R}^d} b^d H(b^2 t, bx)\,\mathrm{d}x = \int_{\mathbb{R}^d} H(b^2 t, y)\,\mathrm{d}y.$$

Since $H(1,x)$ is rotationally invariant, it depends only on $|x|$. Let φ be the function so that $H(1,x) = \varphi(|x|)$. Therefore choosing $b = t^{-1/2}$ in (3.7), we get

$$H(t,x) = \frac{1}{t^{d/2}}\varphi\left(\frac{|x|}{\sqrt{t}}\right).$$

This formulation summarizes the invariance conditions of H. We will now plug it into the equation to compute the formula of φ for which this is a solution to the heat equation. We compute

$$H_t = -\frac{d}{2}t^{-d/2-1}\varphi\left(\frac{|x|}{\sqrt{t}}\right) - \frac{1}{2}t^{-d/2-3/2}|x|\varphi'\left(\frac{|x|}{\sqrt{t}}\right),$$
$$\Delta H = t^{-d/2-1}\left(\frac{d-1}{|x|/\sqrt{t}}\cdot\varphi'\left(\frac{|x|}{\sqrt{t}}\right) + \varphi''\left(\frac{|x|}{\sqrt{t}}\right)\right).$$

Let $r = |x|/\sqrt{t}$. Replacing in the previous formulas we get

$$H_t = t^{-d/2-1}\left(-\frac{d}{2}\varphi(r) - \frac{1}{2}r\varphi'(r)\right),$$
$$\Delta H = t^{-d/2-1}\left(\frac{d-1}{r}\varphi'(r) + \varphi''(r)\right).$$

From the equation $H_t = \Delta H$, we deduce,

$$-\frac{d}{2}\varphi(r) - \frac{1}{2}r\varphi'(r) = \frac{d-1}{r}\varphi'(r) + \varphi''(r). \tag{3.8}$$

We make the following two observations

$$\frac{d}{2}\varphi(r) + \frac{1}{2}r\varphi'(r) = r^{-d+1}\partial_r\left(\frac{r^d}{2}\varphi(r)\right),$$
$$\frac{d-1}{r}\varphi'(r) + \varphi''(r) = r^{-d+1}\partial_r\left(r^{d-1}\varphi'(r)\right).$$

Replacing these two identities into (3.8), we get

$$\partial_r\left(\frac{r^d}{2}\varphi(r) + r^{d-1}\varphi'(r)\right) = 0.$$

Therefore, this function that we are differentiating must be a constant. Since it vanishes at $r = 0$, it must be zero:

$$\frac{r^d}{2}\varphi(r) + r^{d-1}\varphi'(r) = 0$$

Canceling out the common factor r^{d-1},

$$\frac{r}{2}\varphi(r) + \varphi'(r) = 0$$

It follows that

$$\varphi(r) = ce^{-r^2/4},$$

for some constant c. We choose this constant c to enforce the integral equal to one condition (3.6). This constant c can be computed to be $c = (4\pi)^{-d/2}$.

Therefore, we arrived at the explicit form of the fundamental solution

$$H(t,x) = \frac{1}{(4\pi t)^{d/2}}e^{-|x|^2/4t}. \tag{3.9}$$

For all $t > 0$ the function $H(t,x)$ is C^∞ in x and decays very quickly as $|x| \to \infty$.

3.3.3 A formula for the solution of the initial value problem in the whole space

In this section, we use the fundamental solution H to obtain a formula for the solution of the heat equation for arbitrary initial data in the full space. For this, it is essential to understand the behavior of $H(t,x)$ as $t \to 0$. The formula (3.9) does not make sense for $t = 0$ since it involves a division by zero.

For different values of t, the function $H(t,\cdot)$ is a different rescaling of the profile $\varphi(|x|)$. Its integral is constant equal to one. The following lemma shows that in some sense the mass of $H(t,\cdot)$ concentrates around the origin as $t \to 0$.

Lemma 3.3.1. *For any $\delta > 0$, the following holds*

$$\lim_{t\to 0}\int_{\mathbb{R}^d\setminus B_\delta} H(t,x)\,\mathrm{d}x = 0.$$

Consequently

$$\lim_{t\to 0}\int_{B_\delta} H(t,x)\,\mathrm{d}x = 1.$$

Proof. Recall that $H(t,x) = t^{-d/2}\varphi(|x|/\sqrt{t})$. We make the change of variables $y = x/\sqrt{t}$. Noting that $\mathrm{d}y = t^{-d/2}\,\mathrm{d}x$, we obtain

$$\int_{\mathbb{R}^d \setminus B_\delta} H(t,x)\,\mathrm{d}x = \int_{\mathbb{R}^d \setminus B_{\frac{\delta}{\sqrt{t}}}} \varphi(|y|)\,\mathrm{d}y$$

As $t \to 0$, the radius of the ball $\delta/\sqrt{t}$ goes to infinity. We are integrating the tails of the integrable function $\varphi(|x|) = H(1,x)$, which clearly goes to zero. This concludes the first part of the lemma.

The second part of the lemma follows as a consequence of the first part since

$$\lim_{t\to 0} \int_{B_\delta} H(t,x)\,\mathrm{d}x = \int_{\mathbb{R}^d} H(t,x)\,\mathrm{d}x - \lim_{t\to 0}\int_{\mathbb{R}^d\setminus B_\delta} H(t,x)\,\mathrm{d}x = 1.$$

□

Theorem 3.3.2. *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a bounded continuous function. Then the function*

$$u(t,x) = \int_{\mathbb{R}^d} f(y) H(t, x-y)\,\mathrm{d}y, \tag{3.10}$$

is the unique solution of the initial value problem

$$\begin{aligned} u_t - \Delta u &= 0 \quad \text{in } (0,\infty)\times\mathbb{R}^d, \\ u(0,x) &= f(x). \end{aligned}$$

Proof. We proved before that the equation can have at most one bounded solution. We must verify now that the function u given by the formula above is indeed a solution.

We first check that this function u satisfies the equation. We compute

$$\begin{aligned} u_t(t,x) &= \int_{\mathbb{R}^d} f(y) H_t(t,x-y)\,\mathrm{d}y, \\ \Delta u(t,x) &= \int_{\mathbb{R}^d} f(y)\Delta H(t,x-y)\,\mathrm{d}y. \end{aligned}$$

Since $H_t = \Delta H$, the two integrands coincide and $u_t = \Delta u$. We remark here that we can commute integration with differentiation for all bounded functions f in this case because H is C^∞ and it decays very fast at infinity together with its derivatives.

We must now check the initial condition $u(0,x) = f(x)$. The formula (3.10) does not make sense literally for $t = 0$ since the function $H(0,x)$ is not well defined (it is in some sense a Dirac delta). The appropriate way to understand the initial condition is as a limit for $t \to 0$. Thus, we will prove that

$$\lim_{t\to 0}\int_{\mathbb{R}^d} f(y) H(t,x-y)\,\mathrm{d}y = f(x). \tag{3.11}$$

We can prove this using Lemma 3.3.1 and the continuity of f. Indeed, for every $\varepsilon > 0$ and $x \in \mathbb{R}^d$, there exists a $\delta > 0$ so that $f(x) - \varepsilon < f(y) < f(x) + \varepsilon$ if $|y - x| < \delta$. Therefore

$$\begin{aligned} \int_{\mathbb{R}^d} f(y)H(t,x-y)\,\mathrm{d}y &= \int_{B_\delta(x)} f(y)H(t,x-y)\,\mathrm{d}y + \int_{\mathbb{R}^d\setminus B_\delta(x)} f(y)H(t,x-y)\,\mathrm{d}y, \\ &\le (f(x)+\varepsilon)\int_{B_\delta} H(t,y)\,\mathrm{d}y + (\max f)\int_{\mathbb{R}^d\setminus B_\delta} H(t,y)\,\mathrm{d}y. \end{aligned}$$

Similarly

$$\int_{\mathbb{R}^d} f(y)H(t,x-y)\,\mathrm{d}y \ge (f(x)-\varepsilon)\int_{B_\delta} H(t,y)\,\mathrm{d}y + (\min f)\int_{\mathbb{R}^d\setminus B_\delta} H(t,y)\,\mathrm{d}y.$$

Taking lim sup and lim inf and using Lemma 3.3.1,

$$\limsup_{t\to 0} \int_{\mathbb{R}^d} f(y)H(t,x-y)\,\mathrm{d}y \le f(x)+\varepsilon,$$
$$\liminf_{t\to 0} \int_{\mathbb{R}^d} f(y)H(t,x-y)\,\mathrm{d}y \ge f(x)-\varepsilon.$$

Since ε is arbitrary, we conclude the proof of (3.11). □

Some remarks can be made about the previous proof. We observe that if f is uniformly continuous, then the convergence in (3.11) is uniform. Since every continuous function is uniformly continuous in any compact set, we can see that the convergence in (3.11) is uniform in any ball $\overline{B}_R$ for any arbitrary $R < +\infty$ assuming only that f is continuous and bounded.

The formula (3.10) makes sense even if f is not continuous. For any bounded f (and also for some unbounded ones), this formula gives us a function u which is C^∞ and solves the heat equation as soon as $t > 0$. Our proof above says that the initial value $u(0,x) = f(x)$ holds at any point x where f is continuous. When the initial value function f is very rough, the identity $\lim_{t\to 0} u(t,x) = f(x)$ is more delicate and needs to be interpreted appropriately. For all practical purposes, we can take for granted that the formula (3.9) always gives us the solution to the heat equation. Moreover, no matter how rough the initial function f is, the solution u is always C^∞ smooth as soon as $t > 0$.

The Dirac delta

Looking at the formula for $H(t,x)$ and the result of Lemma 3.3.1, we see that as $t \to 0$, the whole distribution of $H(t,x)$ is concentrating around $x = 0$. As $t \to 0$, $H(t,x)$ does not converge to any function. It converges to the Dirac delta δ_0 which is a measure concentrated in the point $x = 0$.

The Dirac delta δ_0 is not a function. It is a formal object which follows the following integration rule:

$$\int_{\mathbb{R}^d} f(x)\delta_0(x)\,\mathrm{d}x = f(0).$$

When we say that $H(t,x) \to \delta_0(x)$ as $t \to 0$, it is simply a way to say that for every bounded continuous function f,

$$\lim_{t\to 0} \int_{\mathbb{R}^d} f(x)H(t,x)\,\mathrm{d}x = \int_{\mathbb{R}^d} f(x)\delta_0(x)\,\mathrm{d}x = f(0).$$

We have already proved this in Theorem (3.3.2).

It is sometimes useful to keep the following convolution property in mind

$$\int_{\mathbb{R}^d} f(y)\delta_0(x-y)\,\mathrm{d}y = f(x).$$

In one dimension, the Dirac delta δ_0 is the derivative of the heaviside function

$$H(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x \le 0. \end{cases}$$

The identity $\delta_0 = H'$ is formally justified by the *integration by parts* formula

$$\int_{\mathbb{R}} f(x)\delta_0(x)\,\mathrm{d}x = -\int_{\mathbb{R}} f'(x)H(x)\,\mathrm{d}x.$$

which holds for all f differentiable and compactly supported.

3.3.4 The inhomogeneous heat equation

The purpose of this section is to develop a formula to solve the problem

$$\begin{aligned} u_t - \Delta u &= g(t,x) \quad \text{for } x \in \mathbb{R}^d \text{ and } t > 0, \\ u(0,x) &= f(x). \end{aligned} \tag{3.12}$$

The solution u will be the sum of two functions $u = u_1 + u_2$, each solving a simpler problem.

$$\begin{aligned} \partial_t u_1 - \Delta u_1 &= 0 \quad \text{for } x \in \mathbb{R}^d \text{ and } t > 0, \\ u_1(0,x) &= f(x). \end{aligned}$$

and

$$\begin{aligned} \partial_t u_2 - \Delta u_2 &= g(t,x) \quad \text{for } x \in \mathbb{R}^d \text{ and } t > 0, \\ u_2(0,x) &= 0. \end{aligned} \tag{3.13}$$

Theorem (3.3.2) gives us a formula for u_1,

$$u_1(t,x) = \int_{\mathbb{R}^d} f(y) H(t, x-y) \,\mathrm{d}y.$$

We need to find a formula for u_2.

Let us try to guess the right formula for u_2. Imagine that $g(t,x)$ was completely concentrated at a time $t = s$. If we think of the heat equation modeling the evolution of temperature in certain body, the right hand side $g(t,x)$ corresponds to external sources of heat. Having $g(t,x)$ concentrated at a time $t = s$ is like having an instantaneous injection of heat at this specific time. One way to model this is to think of a right hand side of the form $g(t,x) = g(x)\delta_0(t-s)$. The solution to the problem

$$\begin{aligned} v_t - \Delta v &= g(x)\delta_0(t-s), \\ v(0,x) &= 0 \end{aligned}$$

would be equal to zero for all $t < s$, but then it would have to jump suddenly to $u(s,x) = g(s)$. Then it would continue as a regular solution of the heat equation with zero right hand side for $t > s$. The function v would be

$$v(t,x) = \begin{cases} 0 & \text{if } t < s, \\ \int_{\mathbb{R}^d} g(s,y) H(t-s, x-y) \,\mathrm{d}y & \text{if } t > s. \end{cases}$$

Since $g(t,x) = \int_0^\infty g(s,x)\delta_0(t-s) \,\mathrm{d}s$, we should be able to obtain the formula for the solution u_2 of (3.13) by the following integral formula

$$u_2(t,x) = \int_0^t \int_{\mathbb{R}^d} g(s,y) H(t-s, x-y) \,\mathrm{d}y \,\mathrm{d}s. \tag{3.14}$$

The justification above was not meant to be rigorous. We verify the formula in the next proposition.

Proposition 3.3.3. *Assume g is continuous. The formula* (3.14) *indeed gives a solution to* (3.13).

Proof. It is clear that from this formula $u_2(0,x) = 0$. We need to check that the partial differential equation is satisfied.

As usual, we can compute Δu_2 by differentiating inside the integral.

$$\Delta u_2(t,x) = \int_0^t \int_{\mathbb{R}^d} g(s,y) \Delta H(t-s, x-y) \,\mathrm{d}y \,\mathrm{d}s. \tag{3.15}$$

We have to be more careful when we compute $\partial_t u_2$ because the domain of integration depends on t. Consider the following expression with double variables.

$$U(t_1, t_2, x) = \int_0^{t_1} \int_{\mathbb{R}^d} g(s,y) H(t_2 - s, x-y) \,\mathrm{d}y \,\mathrm{d}s.$$

Certainly, $u_2(t,x) = U(t,t,x)$. Therefore, $\partial_t u_2(t,x) = \partial_{t_1} U(t,t,x) + \partial_{t_2} U(t,t,x)$. For $\partial_{t_2} U(t,t,x)$ we can differentiate inside the integral as usual.

$$\partial_{t_2} U(t,t,x) = \int_0^t \int_{\mathbb{R}^d} g(s,y) H_t(t-s, x-y) \, \mathrm{d}y \, \mathrm{d}s.$$

For $\partial_{t_1} U(t,t,x)$, we use the fundamental theorem of calculus.

$$\begin{aligned} \partial_{t_1} U(t,t,x) &= \lim_{s \to t} \int_{\mathbb{R}^d} g(s,y) H(t-s, x-y) \, \mathrm{d}y, \\ &= \lim_{s \to t} \int_{\mathbb{R}^d} g(t,y) H(t-s,x-y) \, \mathrm{d}y + \int_{\mathbb{R}^d} (g(s,y) - g(t,y)) H(t-s,x-y) \, \mathrm{d}y \end{aligned}$$

Recall that the heat kernel $H(t,x)$ is not well defined at $t = 0$. That is the reason why we write the limit $s \to t$ instead of simply evaluating at $s = t$.

The first term converges to $g(t,x)$ because of Theorem 3.3.2. The second term converges to zero because of the continuity of g. Thus, $\partial_{t_1} U(t,t,x) = g(t,x)$. Since $\partial_t u_2 = \partial_{t_1} U(t,t,x) + \partial_{t_2} U(t,t,x)$, we get

$$\partial_t u_2 = \int_0^t \int_{\mathbb{R}^d} g(s,y) H_t(t-s,x-y) \, \mathrm{d}y \, \mathrm{d}s + g(t,x).$$

Combining this expression with (3.15), we finish the proof. □

We can now combine the formulas for u_1 and u_2 to derive the final formula for the solution u of the inhomogeneous problem (3.12). We get

$$u(t,x) = \int_{\mathbb{R}^d} f(y) H(t, x-y) \, \mathrm{d}y + \int_0^t \int_{\mathbb{R}^d} g(s,y) H(t-s, x-y) \, \mathrm{d}y \, \mathrm{d}s. \tag{3.16}$$

3.3.5 Duhamel principle

There is a general principle to derive a formula to solve linear evolution equations with a non-zero right hand side, in terms of the solution to the initial value problem with zero right hand side. Above, we did it in the particular case of the heat equation. Here, we explain the general method in an abstract setting.

Let us suppose we have an initial value problem

$$\begin{aligned} u_t - \mathcal{A}u &= 0, \\ u(0,\cdot) &= f. \end{aligned}$$

Here $\mathcal{A}$ refers to an abstract operator. It was $\mathcal{A}u = \Delta u$ for the heat equation. It would be $\mathcal{A}u = b \cdot \nabla_x u$ in the case of the transport equation. In these two cases (and in general for all linear PDEs) $\mathcal{A}$ is an operator that maps a function into another. The method applies likewise when $\mathcal{A}$ is a linear operator from $\mathbb{R}^d$ to itself and the equation above is an ODE.

Let us suppose we know how to solve this initial value problem and let us call S_t its corresponding *solution* operator that maps f into $u(t,\cdot)$ for each value of $t > 0$. In the case of the heat equation we would have

$$S_t f(x) = \int_{\mathbb{R}^d} f(y) H(t, x-y) \, \mathrm{d}y.$$

For other equations, the operator S_t would consist of a different formula.

We now consider the problem with a non-zero right hand side.

$$\begin{aligned} u_t - \mathcal{A}u &= g, \\ u(0,\cdot) &= f \end{aligned}$$

Then, the solution will be given by

$$u(t,\cdot) = S_t f + \int_0^t S_{t-s} g(s,\cdot) \, \mathrm{d}s. \tag{3.17}$$

This is known as Duhamel principle. It can be verified directly by plugging the formula (3.17) into the equation above. In the case of the heat equation, and (3.17) becomes (3.16).

As we mentioned above, the method works in great generality, including also the simpler ODE setting. If we consider the ODE: $u(t) \in \mathbb{R}^d$ with $u'(t) - Au(t) = g(t)$, then the solution operator $S_t : \mathbb{R}^d \to \mathbb{R}^d$ would correspond to $S_t u = \exp(tA)u$ and (3.17) becomes the well known formula from ODEs

$$u(t) = \exp(tA)u_0 + \int_0^t \exp((t-s)A)g(s) \, \mathrm{d}s.$$

In the extreme case that $u(t) \in \mathbb{R}$ and $\mathcal{A} = 0$, the formula (3.17) corresponds simply to the fundamental theorem of calculus.

Later, we will use the formula (3.17) in the context of the wave equation.

3.4 Random walks and numerical analysis

We give a simple example of a problem in random walks whose solution is approximated using the heat equation. We also use this model to motivate the numerical method of finite differences.

3.4.1 Random walks

Consider a sequence of cells, numbered from 1 to $N-1$, and a particle that moves from its current cell to either the one on the right or the one of the left every second. The choice of which direction to move is random, with equal probability, and independent of all previous choices. We want to compute the probability that the particle moves T times (T seconds) without exiting from one of the ends of the row of cells. [1]

At first sight, this problem may seem to have very little to do with the heat equation. In the heat equation we look for a function u depending on a continuous space variable x and time t. Here we have two discrete variables. The particle's position, which is the index of the current cell, is a discrete space variable. The number of seconds is like a discrete time variable. When the length of the row is very large and the number of seconds is also large, we will see that this problem is a good approximation of the heat equation.

Let $p_{i,j}$ be the probability that a particle at the cell number j will last at least i steps without exiting through the ends.

If $i = 0$, there are no more steps to take, and the probability is therefore equal to one for all values of $j = 1, \dots, N-1$. That is

$$p_{0,j} = 1 \quad \text{for } j = 1, \dots, N-1. \tag{3.18}$$

The event of the particle exiting at the end points can be modeled as the index reaching 0 or N. In that case we get

$$p_{i,j} = 0 \quad \text{for } j = 0 \text{ or } N. \tag{3.19}$$

For all other values of i and j, an extra step will be made. After this step, the particle will land either in cell $j-1$ or $j+1$ with equal likelihood, and one fewer step will be left. This is encoded in the recursive formula.

$$p_{i+1,j} = \frac{1}{2}\left(p_{i,j-1} + p_{i,j+1}\right). \tag{3.20}$$

Let u be the solution to the following heat equation

$$\begin{aligned} u(0,x) &= 1 && \text{for } x \in (0,1), \\ u(t,0) = u(t,1) &= 0 && \text{for all } t > 0, \\ u_t &= u_{xx}/2 && \text{for } (t,x) \in (0,\infty) \times (0,1). \end{aligned}$$

From the values of u, we will approximate the values of $p_{i,j}$. Indeed, let $h = 1/N$ and

$$q_{i,j} = u(h^2 i, hj).$$

[1] There are several possible contextualizations for this model, including the probability that a drunk man would not fall from the roof. Others are left to your imagination.

Let us fix $x = hj$ and consider the Taylor expansion of u with respect to the t variable around $t = h^2 i$. The following is the expansion of order one including the error term.

$$u(t, hj) = u(h^2 i, hj) + u_t(h^2 i, hj)(t - h^2 i) + \frac{1}{2} u_{tt}(s, hj)(t - h^2 j)^2.$$

Here s is some intermediate value in the interval $(h^2 j, t)$. Applying the formula above to $t = h^2(i+1)$ we get

$$u(h^2(i+1), hj) = u(h^2 i, hj) + u_t(h^2 i, hj) h^2 + O(h^4).$$

The last term $O(h^4)$ means that there is an extra error term which is bounded by Ch^4 for some constant C (which in this case depends on the maximum value of u_{tt}).

Rearranging the terms, we obtain

$$u_t(h^2 i, hj) = \frac{u(h^2(i+1), hj) - u(h^2 i, hj)}{h^2} + O(h^2),$$

which is the same as

$$u_t(h^2 i, hj) = \frac{q_{i+1,j} - q_{i,j}}{h^2} + O(h^2).$$

An analogous computation using the fourth order Taylor expansion in the x variable gives us the expression

$$u_{xx}(h^2 i, hj) = \frac{u(h^2 i, (j+1)) - 2u(h^2 i, hj) + u(h^2 i, h(j-1))}{h^2} + O(h^2).$$

This is the same as

$$u_{xx}(h^2 i, hj) = \frac{q_{i,j+1} - 2q_{i,j} + q_{i,j-1}}{h^2} + O(h^2).$$

From the equation, $u_t = u_{xx}/2$, we see that

$$q_{i+1,j} - q_{i,j} = \frac{q_{i,(j+1)} - 2q_{i,j} + q_{i,(j-1)}}{2} + O(h^4).$$

Rearranging the terms, we obtain

$$q_{i+1,j} = \frac{q_{i,(j+1)} + q_{i,(j-1)}}{2} + O(h^4).$$

From the initial condition $u(0, x) = 1$, we deduce that

$$q_{0,j} = 1.$$

From the boundary condition $u(t, 0) = u(t, 1) = 0$, we deduce that

$$q_{i,0} = q_{i,N} = 0.$$

We see that the sequence $q_{i,j}$ satisfy the same initial and boundary conditions (3.18) and (3.19) as $p_{i,j}$. Moreover, it satisfies also (3.20) except for a small error of order $O(h^4)$. Naturally, the values of $q_{i,j}$ and $p_{i,j}$ will differ by very little when h is small (which corresponds to N being large).

Remark 3.4.1. The error terms $O(h^4)$ depend on the value of u_{tt} and u_{xxxx} in a neighborhood of each point. A precise estimation of the total error is more difficult than usual for this particular example because these quantities go to infinity at the corners $(0, 0)$ and $(1, 0)$ because of the discontinuous boundary condition.

3.4.2 The method of finite differences

We now describe the reverse process of the previous section. We start with a partial differential equation (for example the heat equation), and we want to devise a discrete computation that will approximate the values of its solution. In the case of the heat equation $u_t = u_{xx}/2$, we could approximate the solution using the scheme (3.20) for a small value of h. For arbitrary equations, we would follow the method described below.

We want to approximate the value of certain function $u(t,x)$ satisfying certain PDE. We write

$$u_{i,j} = u(ki, hj).$$

The values of k and h are arbitrary small numbers. The accuracy of the computation tends to improve the smaller h and k are. But this also makes the computation much more time consuming.

Virtually any PDE can be transformed into a recurrence relation for the values of $u_{i,j}$ replacing the derivatives of u with their corresponding incremental quotient approximation.

$$\begin{aligned} u_t(ki,hj) &\approx \frac{u_{i+1,j} - u_{i,j}}{k}, \\ u_x(ki,hj) &\approx \frac{u_{i,j+1} - u_{i,j}}{h}, \\ u_{xx}(ki,hj) &\approx \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2}. \end{aligned}$$

For example, we could try to approximate the solution of a complicated equation like

$$u_t = u_{xx} + u u_x,$$

by the finite difference scheme

$$\frac{u_{i+1,j} - u_{i,j}}{k} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2} + u_{i,j}\left(\frac{u_{i,j+1} - u_{i,j}}{h}\right).$$

This can be rewritten as a more explicit iterative scheme

$$u_{i+1,j} = u_{i,j} + k\left(\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2} + u_{i,j}\left(\frac{u_{i,j+1} - u_{i,j}}{h}\right)\right).$$

The boundary conditions of the equation are translated directly into some given values for $u_{i,j}$.

The method can be easily adapted to equations in arbitrary domains $\Omega \subset \mathbb{R}^d$. Indeed, suppose that $x \in \Omega \subset \mathbb{R}^d$. We would use the indexes $i, j_1, j_2, \dots, j_d$ and

$$u_{i,j_1,j_2,\dots,j_d} = u(ki, hj_1, \dots, hj_d).$$

Partial derivatives of u with respect to each direction x_r correspond to differential quotients with respect to each index j_r.

It cannot be used to obtain an exact solution and it rarely helps in proving theorems about qualitative properties of solutions to PDE. But it is useful to approximate solutions with arbitrary precision and is truly general. There are, however, several things that can go wrong. The finite difference method must be used with care. Some of the dangers of the method are listed below.

- For a complicated equation, it may be impossible to estimate the error in the computation. In that case, we would have no idea how accurate our results are.
- A computation in dimension 3 or more, with an acceptably small value of h, takes an enormous number of points. The computation can be prohibitively long even for a fast computer.
- Each time we compute a value of $u_{i+1,j}$ there is a small error in that computation. Sometimes these errors accumulate and produce a very large error at the end. The finite difference schemes for which this happens are called *unstable*.

- We may be trying to solve the wrong problem. For example, we can write down a finite difference scheme for an equation which does not have a solution. If we run this scheme we will get some numbers, but they will be meaningless.

We would be able to reduce complications of the methods with a further understanding of the theory of partial differential equations. It is possible to write more than one finite difference scheme for the same equation. This is because there is more than one way to approximate derivatives with incremental quotients. For example

$$u_x(ki, hj) \approx \frac{u_{i,j+1} - u_{i,j}}{h} \approx \frac{u_{i,j} - u_{i,j-1}}{h} \approx \frac{u_{i,j+1} - u_{i,j-1}}{2h}.$$

It can happen that a finite difference schemes is unstable, while another one for the same equation is stable. The study of how to do these approximations correctly and how to precisely estimate the computational error is part of advanced numerical analysis.

3.5 Exercises

Exercise 3.1. *Let $\Omega \subset \mathbb{R}^d$ be open, bounded and with $\partial\Omega$ piecewise C^1. Let $u : [0,\infty) \times \overline{\Omega} \to \mathbb{R}$ be a solution of the heat equation with Neumann boundary conditions,*

$$u_t - \Delta u = 0 \text{ in } (0,\infty) \times \Omega,$$
$$u_\nu = 0 \text{ on } \partial\Omega.$$

(a) Prove that for any convex function $F : \mathbb{R} \to \mathbb{R}$, the quantity

$$H(t) := \int_\Omega F(u(t,x)) \, \mathrm{d}x,$$

is nonincreasing.

(b) Prove that the following quantity is nonincreasing.

$$H(t) := \int_\Omega |\nabla u(t,x)|^2 \, \mathrm{d}x.$$

Note. A function $f : \mathbb{R} \to \mathbb{R}$ is convex when $f'' \geq 0$ everywhere.
Hint. For part (b), prove that $H'(t) = -2\int |\Delta u|^2$.

Exercise 3.2. *Let $u : [0,\infty) \times \Omega \to \mathbb{R}$ be a solution of*

$$u_t - \Delta u + u^3 = 0 \text{ in } (0,\infty) \times \Omega,$$
$$u_\nu = 0 \text{ on } \partial\Omega.$$

Prove that the quantity

$$H(t) := \int_\Omega \frac{|\nabla u|^2}{2} + \frac{u^4}{4} \, \mathrm{d}x,$$

is non-increasing.

Exercise 3.3. *When we solved the heat equation with Dirichlet boundary conditions by the method of separation of variables, we ended up with an expression like*

$$u(t,x) = \int_0^1 f(y) H(t,x,y) \, \mathrm{d}y,$$

where H is the explicit function given by

$$H(t,x,y) = 2\sum_{k=1}^{\infty} \sin(\pi k x)\sin(\pi k y)\exp(-\pi^2 k^2 t).$$

Moreover, H solves the heat equation as a function of (t,x) and also as a function of (t,y).

Let us accept that this formula is correct and solves the heat equation for any initial value $f(x)$.

(a) Prove that $H(t, x, y) \geq 0$ *for any* $t > 0$, $x \in (0, 1)$ *and* $y \in (0, 1)$.

(b) Prove that for any $t > 0$ *and* $x \in (0, 1)$,

$$\int_0^1 H(t, x, y) \, \mathrm{d}y \leq 1.$$

Hint. This problem can be solved without looking at the explicit formula for H. Use the maximum principle.

Exercise 3.4. *Let* $f_n : \mathbb{R}^d \to \mathbb{R}$ *be a sequence of bounded continuous functions that converge uniformly to another function* f. *Let* $u_n : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ *be the corresponding sequence of bounded solutions to the heat equation with initial value* f_n*:*

$$u_n(0, x) = f_n(x),$$
$$\partial_t u_n - \Delta u_n = 0 \text{ in } (0, \infty) \times \mathbb{R}^d.$$

Prove that $u_n \to u$ *uniformly, where* u *is the solution of the heat equation with initial value* f.

Note. The previous question shows some form of continuous dependence of the solution u with respect to its initial data f.

Exercise 3.5. *Let* $u : \overline{\Omega_T} \to \mathbb{R}$ *be continuous on* $\overline{\Omega}_T$ *and solve*

$$u_t + b(t, x) \cdot \nabla u = \Delta u \qquad \text{in } \Omega_T.$$

Here $b(t, x)$ *is an arbitrary continuous vector field.*

Prove that u *satisfies the maximum principle, that is*

$$\max_{\partial_p \Omega_T} u = \max_{\overline{\Omega}_T} u,$$
$$\min_{\partial_p \Omega_T} u = \min_{\overline{\Omega}_T} u.$$

Recall our notation $\Omega \subset \mathbb{R}^n$ is a bounded open set, $\Omega_T = (0, T] \times \Omega$ and $\partial_p \Omega_T = (\{0\} \times \Omega) \cup ([0, T] \times \partial\Omega)$.

Exercise 3.6. *Let* Ω *be an bounded open set in* $\mathbb{R}^d$ *with a* C^2 *boundary. Prove the maximum principle for the heat equation with Neumann boundary conditions. That is, for any function* $u : [0, T] \times \overline{\Omega} \to \mathbb{R}$ *which is smooth and solves*

$$u_t - \Delta u = 0 \text{ in } (0, T] \times \Omega,$$
$$u_\nu = 0 \text{ on } (0, T] \times \partial\Omega.$$

Then

$$\max_{[0,T] \times \overline{\Omega}} u = \max_{x \in \overline{\Omega}} u(0, x).$$

Hint. Pick any C^2 function $\varphi : \overline{\Omega} \to \mathbb{R}$ that vanishes on $\partial\Omega$ for which $\varphi_\nu < 0$ everywhere on $\partial\Omega$. For example, you can take $\varphi(x)$ equal to the distance between x and $\partial\Omega$ when x is sufficiently close to $\partial\Omega$ (the distance function is not C^2 everywhere, but only near the boundary). Then, consider the function $v(t, x) = u(t, x) + \varepsilon(\varphi(x) - Ct)$, for a suitable constant C.

Exercise 3.7. *Let* $u : [0, T] \times \overline{B_1} \to \mathbb{R}$ *be a continuous function that satisfies*

$$u_t - \Delta u + u^2 = 0, \qquad \text{in } (0, T] \times B_1,$$
$$u(t, x) = 0 \qquad \text{on } (0, T] \times \partial B_1.$$

Prove that $u(T, x) \leq 1/T$ *(regardless of the initial data).*

Exercise 3.8. *Write down the explicit solutions to the heat equation*

$$\begin{aligned} u_t - \Delta u = 0 \text{ in } (0,\infty) \times \mathbb{R}^d, \\ u(0,x) = f(x), \end{aligned}$$

for the following values of f.

(a)

$$f(x) = \prod_{i=1}^{d} \sin(x_i).$$

(b)

$$f(x) = e^{-|x|^2}.$$

(c)

$$f(x) = x_1 \, e^{-|x|^2}.$$

Exercise 3.9. *Let $f : [0,\infty) \to \mathbb{R}$. Write a formula for the solution u of the following equation*

$$\begin{aligned} u_t - u_{xx} = 0 \text{ in } (0,\infty) \times (0,\infty), \\ u(0,x) = f(x), \\ u(t,0) = 0. \end{aligned}$$

Exercise 3.10. *Assume that f is a continuous and compactly supported function in $\mathbb{R}^d$. Let u be the bounded solution to the heat equation in the whole space, with initial value f.*

(a) Prove that for any $x \in \mathbb{R}^d$, $\lim_{t\to\infty} u(t,x) = 0$.

(b) Prove that for any $x \in \mathbb{R}^d$,

$$\lim_{t\to\infty} (4\pi t)^{d/2} u(t,x) = \int_{\mathbb{R}^d} f(y) \, \mathrm{d}y.$$

Exercise 3.11. *Let $u : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ be a solution to the heat equation*

$$u_t = \Delta u.$$

Assume u is bounded and $u(0,x) = f(x)$. Prove that $u(T,x)$ is differentiable in x and

$$\max |\nabla_x u(T,x)| \leq \frac{C}{\sqrt{T}} \max |f|.$$

Hint. Use the fundamental solution of the heat equation and differentiate inside the integral.

Exercise 3.12. *Let $u : (-\infty,0] \times \mathbb{R}^d \to \mathbb{R}$ be a solution to the heat equation*

$$u_t = \Delta u.$$

*We say this solution is **ancient** since it exists starting at time $-\infty$. Prove that if u is bounded then it is constant.*

Hint. Use the result from the previous question.

Exercise∗ 3.13. *Prove that the uniqueness result of Theorem 3.2.3 can be extended to functions with polynomial growth in the following way. Assume $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ satisfies, for some positive constants q and C,*

$$\begin{aligned} u_t - \Delta u = 0 \qquad & \text{in } (0,T] \times \mathbb{R}^d, \\ u(0,x) = 0 \qquad & \text{for } x \in \mathbb{R}^d, \\ u(t,x) \leq C(1+|x|)^q \quad & \text{in } (0,T] \times \mathbb{R}^d. \end{aligned}$$

Prove that $u \equiv 0$.

Exercise 3.14. *Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a non-negative integrable function so that*

$$\int_{\mathbb{R}} \Phi(x)\, \mathrm{d}x = 1.$$

Prove that for any bounded continuous function $u : \mathbb{R} \to \mathbb{R}$, we have

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}} u(x-y) \frac{1}{\varepsilon} \Phi\left(\frac{y}{\varepsilon}\right)\, \mathrm{d}y = u(x).$$

Exercise 3.15. *Let $u, v : \Omega_T \to \mathbb{R}$ be such that $u(t,x) = v(t,x) = 0$ for all $x \in \partial\Omega$. Assume that u satisfies $u_t - \Delta u = f$ and v satisfies $v_t + \Delta v = g$. Prove that if*

$$A(t) := \int_U u(t,x) v(t,x)\, \mathrm{d}x$$

then

$$A'(t) = \int_U f(t,x) v(t,x) + u(t,x) g(t,x)\, \mathrm{d}x.$$

Exercise∗ 3.16 (Local smoothing effect)**.** *Let $u : [-1,0] \times B_1 \to \mathbb{R}$ be a solution to the heat equation $u_t - \Delta u = 0$. Let $\varphi : [-1,0] \times B_1 \to \mathbb{R}$ be a smooth, nonnegative, function such that*

$$\varphi(t,x) = \begin{cases} 1 & \text{if } t \geq -1/4 \text{ and } |x| \leq 1/4, \\ 0 & \text{if } t \leq -1/2 \text{ or } |x| \geq 1/2. \end{cases}$$

Let us define $v(t,x) = \varphi(t,x) H(-t,x)$, where $H(t,x)$ is the heat kernel (in the full space) and $D := ([-1/2,0] \times B_{1/2}) \setminus ([-1/4,0] \times B_{1/4})$

(a) Prove that

$$u(0,0) = \int_D (v_t + \Delta v)\, u(t,x)\, \mathrm{d}x\, \mathrm{d}t.$$

(b) Let $(t,x) \in \mathbb{R}^{1+d}$ and $r > 0$. Assume that u solves the heat equation in a domain containing $(t-r^2, t] \times B_r(x)$. Prove that

$$u(t,x) = r^{-d-2} \int_{(t-r^2,t] \times B_r(x)} \big(v_t((s-t)/r^2, (x-y)/r) + \Delta v((s-t)/r^2, (x-y)/r)\big)\, u(s,y)\, \mathrm{d}y\, \mathrm{d}s.$$

(c) Under the same assumptions as in part (b), prove that

$$\nabla u(t,x) = r^{-d-3} \int_{(t-r^2,t] \times B_r(x)} \big(\nabla v_t((s-t)/r^2, (x-y)/r) + \nabla \Delta v((s-t)/r^2, (x-y)/r)\big)\, u(s,y)\, \mathrm{d}y\, \mathrm{d}s.$$

(d) Prove that there is a constant $C > 0$ so that any function u which solves the equation $u_t - \Delta u = 0$ in $[t_0 - r^2, t_0] \times B_r(x_0)$ satisfies the estimate

$$|\nabla u(t_0,x_0)| \leq \frac{C}{r} \max\left\{|u(t,x)| : (t,x) \in [t_0 - r^2, t_0] \times B_r(x_0)\right\}.$$

(c) Prove that any C^2 function that solves the heat equation in $(0,1) \times B_1$ is actually C^∞ in this domain.

Note. It is best to solve the last exercise only after studying regularity estimates in the easier case of the Laplace equation.

Exercise∗ 3.17. *Let $u : (0,T] \times \overline{\Omega} \to \mathbb{R}$ be a continuous function. Prove that u is smooth and satisfies the heat equation in $(0,T] \times \Omega$ if and only if for any C^2 function v, such that $v = |\nabla v| = 0$ on $\partial\Omega$, if we define*

$$F(t) := \int_\Omega u(t,x) v(t,x)\, \mathrm{d}x,$$

we have

$$F'(t) = \int_\Omega u(t,x)(v_t(t,x) + \Delta v(t,x))\, \mathrm{d}x.$$

Exercise 3.18. *Let $u : [0,T] \times B_1 \to \mathbb{R}$ is a continuous function. Prove that u is smooth and solves the heat equation in $(0,T] \times B_1$ in the following cases.*

(a) Assuming u is C^2 and solves the heat equation in the punctured cylinder $(0,T] \times (B_1 \setminus \{0\})$.

(b) Assuming u is C^2 and solves the heat equation in the two halves of the cylinder $(0,T] \times (B_1 \setminus \{x_1 = 0\})$. Moreover, u is C^1 in the full cylinder $(0,T] \times B_1$.

(c) Give an example of a continuous function that solves the heat equation in the two halves of the cylinder: $(0,T] \times (B_1 \setminus \{x_1 = 0\})$, but it does not solve the heat equation in the full cylinder $(0,T] \times B_1$.

Hint. Naturally, the function for part (c) will not be differentiable on the plane $\{x_1 = 0\}$.

Exercise 3.19. *Analyzing the random walk example, we derived the following iteration scheme*

$$u_{i,j+1} = u_{i,j} + k \left(\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} \right).$$

The solution of the initial value problem for heat equation with Dirichlet boundary conditions was approximated by the formula

$$u(jk, ih) \approx u_{i,j}.$$

The good choices of h and k are not independent. It turns out that k has to be chosen small enough depending on each choice of h. Otherwise, the iteration scheme described above becomes unstable in the sense that it would enhance any small errors and the computed solution would return wild values. An unstable iteration scheme may produce values of $u_{i,j}$ that are strictly larger than the initial values $f_i = u_{i,0}$, dishonoring the maximum principle.

(a) For $h = 0.01$, what would be the maximum value of k so that the iteration scheme is stable? (for example, if $x \in [0,1]$, this value of h corresponds to having 101 points on the mess).

(b) Can you guess a formula for the largest value of k as a function of h for which the scheme is stable?

(c) Justify that for k smaller than the formula you suggested in part (b), a discrete version of the maximum principle holds.

Hint. Go to the website: `http://www.math.uchicago.edu/~luis/pde/fd.html` and play with different values of h and k in "*mesh settings*".

Exercise∗ 3.20 (Backwards uniqueness for the heat equations)**.** *Let u solve the heat equation in $[0,T] \times \mathbb{R}^d$. Let us consider the function*

$$F(t) = \int_{\mathbb{R}^d} u(t,x)^2 \, dx.$$

We assume that u decays at infinity sufficiently fast so that $F(t)$ is well defined.

(a) Prove that F is monotone decreasing.

(b) Prove that F is convex (i.e. $F'' \geq 0$).

(c) Prove that $\log F$ is convex.

(d) Conclude that if $u = 0$ at time $t = T$, then $u \equiv 0$ for all time in $[0,T]$.

Exercise 3.21 (Ill posedness of the backward heat equation)**.** *Prove that there exists a C^∞ bounded function $f : \mathbb{R}^d \to \mathbb{R}$ for which there is no C^2 function $u : [0,1] \times \mathbb{R}^d$ such that*

$$\begin{aligned} u_t - \Delta u &= 0 \qquad \text{in } (0,1) \times \mathbb{R}^d, \\ u(1,x) &= f(x). \end{aligned}$$

This means that the backwards heat equation is not solvable.

Hint. Solve the forward heat equation with a given initial data at time $t = 1/2$ that is not C^∞.

Exercise 3.22. *Let u be a solution to*

$$\begin{aligned} u_t - \Delta u &= 0 && \text{in } (0,T] \times \Omega, \\ u &= 0 && \text{on } [0,T] \times \partial\Omega, \\ u(0,x) &= f(x) && \text{for all } x \in \Omega. \end{aligned}$$

Prove that if $\Delta f \geq 0$ in Ω and $f = 0$ on $\partial\Omega$, then $u_t \geq 0$ in $(0,T] \times \Omega$.

Exercise 3.23. *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a bounded continuous function and $u : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ be the corresponding solution to the heat equation with initial value f. Assume that f has a modulus of continuity ω, i.e. for all $x, y \in \mathbb{R}^d$,*

$$|f(x) - f(y)| \leq \omega(|x-y|).$$

Here $\omega : [0,\infty) \to [0,\infty)$ is a continuous monotone function with $\omega(0) = 0$.

(a) Prove that for all $t > 0$, the function u has the same modulus of continuity in x. That is

$$|u(t,x) - u(t,y)| \leq \omega(|x-y|).$$

(b) Prove that there is another modulus of continuity $\tilde{\omega}$ depending on ω only so that for any $t, h > 0$ and $x \in \mathbb{R}^d$,

$$|u(t+h,x) - u(t,x)| \leq \tilde{\omega}(h).$$

Exercise 3.24. *Let B_1^+ denote the half ball in $\mathbb{R}^d$.*

$$B_1^+ := \{x = (x_1, \dots, x_d) \in \mathbb{R}^d : |x| < 1 \text{ and } x_1 > 0\}.$$

Let $u : [0,T] \times \overline{B_1^+} \to \mathbb{R}$ be a continuous function, which is C^2 in $(0,T] \times B_1^+$ and solves the following heat equation

$$\begin{aligned} u_t - \Delta u &= 0 \text{ in } (0,T] \times B_1^+, \\ u &= 0 \text{ on } [0,T] \times \partial B_1^+, \\ u(0,x) &= f(x) \text{ on } B_1^+, \end{aligned}$$

Let us consider the odd extension $\bar{u} : [0,T] \times \overline{B}_1 \to \mathbb{R}$ given by the formula

$$\bar{u}(t,x) = \begin{cases} u(t,x) & \text{if } x_1 > 0, \\ -u(t,-x_1,x_2,\dots,x_d) & \text{if } x_1 > 0, \\ 0 & \text{if } x_1 = 0. \end{cases}$$

(a) Prove that $\bar{u}$ solves the heat equation

$$\bar{u}_t - \Delta \bar{u} = 0 \text{ in } (0,T] \times B_1.$$

(b) Prove that $\Delta \bar{u} = 0$ on $\{x_1 = 0\}$.

Note. The only difficulty in part (a) is to establish that u is C^2 and solves the equation on $\{x_1 = 0\}$.
Hint. Start from the solution $\tilde{u}$ that solves the heat equation in $(0,T] \times B_1$ with initial value given by the odd extension of f (take for granted that this equation is solvable). Show that this function must vanish on $\{x_1 = 0\}$. Then, use the uniqueness of the heat equation in the half ball to show that $\tilde{u}$ and $\bar{u}$ coincide.

Chapter 4

The Laplace equation

The Laplace equation is satisfied by the stationary states of either the wave or heat equation. It is one of the most fundamental equations of mathematical physics, and the prototypical example of an elliptic equation. The study of the Laplace equation goes hand in hand with the study of harmonic functions. We say a function is *harmonic* in a set Ω when its Laplacian vanishes there. As we will see, harmonic functions have several remarkable properties. They are characterized by their *mean value property*, or alternatively as the minimizers of certain energy.

In this note we provide formulas for solving the Laplace equation with Dirichlet boundary conditions in a ball in arbitrary dimension. We justify the uniqueness of solutions in arbitrary domains by two different techniques: energy methods and the maximum principle. We explore several properties of harmonic functions and we discuss their implications in the context of the Laplace equation.

4.1 Energy methods

Let $\Omega \subset \mathbb{R}^d$ be a bounded open set. We assume that its boundary $\partial\Omega$ is piecewise C^1, so that we can apply the divergence theorem from vector calculus in Ω.

We consider the Dirichlet problem for the Laplace equation. We start analyzing the *homogeneous problem*, that is when the Laplacian of the function u equals zero.

$$\begin{aligned} u &= f \text{ on } \partial\Omega, \\ \Delta u &= 0 \text{ in } \Omega. \end{aligned} \tag{4.1}$$

A solution to (4.1) is a function $u \in C^2(\Omega)$, continuous up to the boundary in $\overline{\Omega}$, such that (4.1) holds.

For the next theorem, we assume that the solution u is C^1 up to the boundary in $\overline{\Omega}$.

Theorem 4.1.1. *From all C^1 functions in $\overline{\Omega}$ that equal f on $\partial\Omega$, the solution u of* (4.1) *is the one that minimizes the quantity:*

$$\int_\Omega |\nabla u|^2 \,\mathrm{d}x. \tag{4.2}$$

Proof. Let v be any other function such that $v = u = f$ on $\partial\Omega$. We compute

$$\begin{aligned} \int_\Omega |\nabla v|^2 \,\mathrm{d}x &= \int_\Omega |\nabla u + \nabla(v-u)|^2 \,\mathrm{d}x, \\ &= \int_\Omega |\nabla u|^2 + 2\nabla u \cdot \nabla(v-u) + |\nabla(v-u)|^2 \,\mathrm{d}x, \end{aligned}$$

Integrating by parts,

$$= \int_\Omega |\nabla u|^2 - 2(v-u)\Delta u + |\nabla(v-u)|^2 \,\mathrm{d}x + 2\int_{\partial\Omega} (v-u)\partial_\nu u \,\mathrm{d}S,$$

The boundary integral is equal to zero because u and v coincide on $\partial\Omega$. Using also that $\Delta u = 0$ in Ω, we get,

$$\begin{aligned} &= \int_\Omega |\nabla u|^2 + |\nabla(v-u)|^2 \, \mathrm{d}x, \\ &\geq \int_\Omega |\nabla u|^2 \, \mathrm{d}x. \end{aligned}$$

□

The regularity assumption $u \in C^1(\overline{\Omega})$ is not strictly necessary for Theorem 4.1.1 to hold, but it is convenient for the proof given above. This minimization principle holds in great generality. It also provides a strategy to prove the existence of solutions to (4.1.1). One can prove that integrals like (4.2) achieve its minimum value at some function u, and then deduce that u solves the equation (4.1). This proof is outside the scope of these notes, because it requires the use of general tools from functional analysis and Sobolev spaces.

Corollary 4.1.2. *If $f = 0$, then $u = 0$ is the only solution of* (4.1).

Proof. Any solution of (4.1) has to minimize the value of the integral (4.2) from all functions so that $u = f$ on $\partial\Omega$. If we set $u = 0$, we get the value of (4.1) to be equal to zero. So, this has to be its minimal value. Moreover, the integral in (4.2) can only be zero if u is a constant. In this case $u = f = 0$ on $\partial\Omega$, so it is forced to be equal to zero. □

Corollary 4.1.3. *For any $g : \Omega \to \mathbb{R}$ and $f : \partial\Omega \to \mathbb{R}$, the in-homogeneous Laplace equation*

$$\begin{aligned} u &= f \text{ on } \partial\Omega, \\ \Delta u &= g \text{ in } \Omega. \end{aligned} \tag{4.3}$$

has at most one solution.

Proof. Let u and v be two solutions of the equation (4.3). Let $w = u - v$. From the linearity of the equation, we get that

$$\begin{aligned} w &= 0 \text{ on } \partial\Omega, \\ \Delta w &= 0 \text{ in } \Omega. \end{aligned}$$

From Corollary 4.1.2, $w = 0$. Thus $u = v$. □

4.2 The maximum principle

In this section we prove a first version of the maximum principle that says that the maximum of a harmonic function u in $\overline{\Omega}$ is achieved on the boundary $\partial\Omega$. Later on, we will also prove the *strong* maximum principle, that says that if the maximum is achieved at an interior point in Ω, then the function u must be constant.

Theorem 4.2.1 (The (weak) maximum principle)**.** *Let $u : \overline{\Omega} \to \mathbb{R}$ be a continuous function so that $\Delta u \geq 0$ in Ω. Then*

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} f.$$

Note that if u is any solution of (4.1), then we can apply Theorem 4.2.1 both to u and $-u$.

The main idea of the proof of the maximum principle is that if the maximum of u is achieved at an interior point $x_0 \in \Omega$, then $D^2 u(x_0) \leq 0$. Therefore, $\Delta u(x_0) \leq 0$. In order to get a contradiction with $\Delta u \geq 0$, we need to somehow make the inequality strict.

Proof. Let $\varepsilon > 0$ be an arbitrarily small number. We will show that the maximum of the function $u(x) + \varepsilon |x|^2$ is achieved on the boundary $\partial\Omega$. This is implies that

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} f + \varepsilon \max_{\overline{\Omega}} |x|^2.$$

The theorem will follow taking $\varepsilon \to 0$.

So, let us consider the function $v(x) = u(x) + \varepsilon|x|^2$ and assume for the sake of contradiction that it achieves its maximum at an interior point $x_0 \in \Omega$. We have that $D^2 v(x_0) \leq 0$ and therefore $\Delta v(x_0) \leq 0$. But then $\Delta u(x_0) = \Delta v(x_0) - \varepsilon 2d < 0$, which is a contradiction with $\Delta u \geq 0$. Thus, the maximum of v must be achieved on the boundary $\partial\Omega$ and we finish the proof. □

The maximum principle gives us an alternative proof of uniqueness for solutions to (4.1).

Corollary 4.2.2. *If $f = 0$, then $u = 0$ is the only solution of* (4.1).

Proof. Since $f = 0$, applying the maximum principle to u and $-u$, we see that necessarily $u = 0$ in $\overline{\Omega}$. □

4.3 The mean value property

Theorem 4.3.1 (Mean value property)**.** *Let $u : \Omega \to \mathbb{R}$ be a C^2 function with $\Delta u = 0$ in Ω. Assume that $B_r(x_0) \subset \Omega$. Then, the following identities hold*

$$u(x_0) = \frac{1}{|\partial B_r|} \int_{\partial B_r(x_0)} u(y)\, \mathrm{d}S = \frac{1}{|B_r|} \int_{B_r(x_0)} u(y)\, \mathrm{d}y.$$

Here, where we write $|B_r|$, we mean the volume of the ball B_r. When we write $|\partial B_r|$, we mean the surface area of the sphere ∂B_r. By integrating over a domain and dividing by its volume, or conversely by integrating over a surface and dividing by its area, we are effectively averaging the function u on those domains.

Proof. Without loss of generality, we set $x_0 = 0$.

We start by analyzing the integral on the sphere, on the right hand side of the first identity. Let

$$F(r) = \frac{1}{|\partial B_r|} \int_{\partial B_r(0)} u(y)\, \mathrm{d}S.$$

Simply by the continuity of u at zero ($= x_0$), we know that

$$\lim_{r \to 0} F(r) = u(0).$$

We are left to show that F is constant in r. For that, we differentiate the formula of F with respect to r.

$$F'(r) = \frac{\mathrm{d}}{\mathrm{d}r} \frac{1}{|\partial B_r|} \int_{\partial B_r(0)} u(y)\, \mathrm{d}S,$$

Changing variables $y \mapsto ry$,

$$= \frac{\mathrm{d}}{\mathrm{d}r} \frac{1}{|\partial B_1|} \int_{\partial B_1} u(ry)\, \mathrm{d}S.$$

Differentiating inside the integral,

$$= \frac{1}{|\partial B_1|} \int_{\partial B_1} y \cdot \nabla u(ry)\, \mathrm{d}S.$$

Changing back variables $ry \mapsto y$,

$$= \frac{1}{|\partial B_r|} \int_{\partial B_r} \left(\frac{y}{r}\right) \cdot \nabla u(y)\, \mathrm{d}S.$$

Noticing $y/r = n$ and applying the divergence theorem,

$$= \frac{1}{|\partial B_r|} \int_{B_r} \Delta u(y) \, \mathrm{d}y = 0.$$

Therefore, $F(r)$ is constant and equal to $u(0)$. This shows the first identity in the theorem. We still have to justify the second.

We write the integral on the ball, using polar coordinates, as an integral over spheres.

$$\frac{1}{|B_r|} \int_{B_r(x_0)} u(y) \, \mathrm{d}y = \frac{1}{|B_r|} \int_0^r \int_{\partial B_s(x_0)} u(y) \, \mathrm{d}S \, \mathrm{d}s,$$

using the identity we have already proved,

$$\begin{aligned} &= \frac{1}{|B_r|} \int_0^r |\partial B_s| u(x_0) \, \mathrm{d}s, \\ &= u(x_0) \frac{1}{|B_r|} \int_0^r |\partial B_s| \, \mathrm{d}s = u(x_0). \end{aligned}$$

This proves the second identity in the theorem and we are done. □

Corollary 4.3.2 (The strong maximum principle)**.** *Let $\Omega \subset \mathbb{R}^d$ be an open and connected set. Let $u : \Omega \to \mathbb{R}$ be harmonic (i.e. $\Delta u = 0$ in Ω). If u achieves its maximum at some interior point in Ω, then u is constant.*

Proof. Let S be the set where the maximum of u is achieved.

$$S = \{x \in \Omega : u(x) = \sup_{\Omega} u\}.$$

We are assuming that S is not empty. From the continuity of u, we know S is closed relative to Ω. Since Ω is connected, if we show that S is open, then we get $S = \Omega$ and u is constant.

So, we are left to show that S is open. If $x \in S$ and $B_r(x) \subset \Omega$, we have, from the mean value property (Theorem 4.3.1) that

$$u(x) = \frac{1}{|B_r|} \int_{B_r(x)} u(y) \, \mathrm{d}y.$$

Since u achieves its maximum at x, then u can be equal to its average on $B_r(x)$ only if u is identically equal to its average on the full ball $B_r(x)$. Otherwise, any smaller value of u would decrease its average. Thus $B_r(x) \subset S$, which means that any point $x \in S$ is interior and S is open. This finishes the proof. □

We consider a function $\varphi : \mathbb{R}^d \to \mathbb{R}$ with the following properties.

- It is radially symmetric: $\varphi(x) = \tilde{\varphi}(|x|)$.
- It is supported in the ball of radius r centered at the origin: $\operatorname{supp} \varphi \subset B_r$.
- Its integral is equal to one: $\int \varphi(x) \, \mathrm{d}x = 1$.

We define the convolution $u * \varphi$ to be the function given by the following formula

$$u * \varphi(x) = \int u(y)\varphi(x-y) \, \mathrm{d}y = \int u(x-y)\varphi(y) \, \mathrm{d}y.$$

Note that the two integrals above have the same value because of the change of variables $y \mapsto x - y$. If $u : \Omega \to \mathbb{R}$, then $u * \varphi(x)$ is well defined provided that $\varphi(x-y) = 0$ whenever $y \notin \Omega$. This holds in the set $\Omega_r := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > r\}$. Thus, $u * \varphi : \Omega_r \to \mathbb{R}$.

Proposition 4.3.3. *Let φ be a function with the properties described above. Assume that $\Delta u = 0$ in Ω. Then, for all $x \in \Omega_r$,*

$$u(x) = u * \varphi(x).$$

Proof. Without loss of generality, we prove the identity assuming that $x = 0$ in order to keep the formulas cleaner. We have

$$u * \varphi(0) = \int_{B_r} u(y)\varphi(-y) \, \mathrm{d}y,$$

using polar coordinates,

$$= \int_0^r \int_{\partial B_s} u(y)\tilde{\varphi}(s) \, \mathrm{d}S \, \mathrm{d}s,$$

applying the mean value property for spheres,

$$= u(0) \int_0^r \int_{\partial B_s} \tilde{\varphi}(s) \, \mathrm{d}S \, \mathrm{d}s = u(0) \left(\int \varphi(y) \, \mathrm{d}y \right) = u(0).$$

□

Theorem 4.3.4. *Let u be a harmonic function in a ball B_r, then $u \in C^\infty$ in the interior of the ball.*

Proof. It turns out that the convolution of any continuous function u with a C^∞ function φ is C^∞. This is because of the formula

$$\partial_{x_i}[u * \varphi] = u * (\partial_{x_i}\varphi).$$

This property follows directly by differentiating inside the integral in the definition of convolutions. Iterating this property we can compute as many derivatives as we want for $u * \varphi$ in terms of an integral involving u and derivatives of φ.

Harmonic functions u coincide with $u * \varphi$ for functions φ with the properties listed above. Thus, any harmonic function is necessarily C^∞. □

Let us fix a function φ_1 as above, which is supported in B_1. We scale it to generate a one-parameter family of functions φ_r which are supported in a ball of radius B_r.

$$\varphi_r(x) = r^{-d}\varphi_1(r^{-1}x).$$

The purpose of the factor r^{-d} up front is to keep $\int \varphi_r \, \mathrm{d}x = 1$.

According to Proposition 4.3.3, if $\Delta u = 0$ on $B_r(x)$, we have $u(x) = u * \varphi_r(x)$. That is

$$u(x) = \int_{\mathbb{R}^d} \varphi_r(x-y)u(y) \, \mathrm{d}y. \tag{4.4}$$

Note that the value of $u(y)$ outside of $B_r(x)$ is irrelevant for the value of the integral above since $\varphi(x-y) = 0$ at those points. Thus, we write it as an integral in the whole space $\mathbb{R}^d$ even if u is not a priori defined outside $B_r(x)$.

We differentiate (4.4) to prove the following Proposition.

Proposition 4.3.5 (Regularity estimates)**.** *If $\Delta u = 0$ in $B_r(x)$, then $|\nabla u(x)| \leq C/r \max_{B_r(x)} |u|$, for a constant C that is independent of u and r. Moreover, there are estimates for all its derivatives*

$$\max_{B_{r/2}} |D^k u| \leq \frac{C}{r^k} \max_{B_r} |u|.$$

Here C is a constant that depends on dimension and the order of differentiation k.

Proof. We differentiate (4.4) and obtain

$$\nabla u(x) = \int_{\mathbb{R}^d} \nabla\varphi_r(x-y)u(y) \, \mathrm{d}y.$$

Using that $\nabla\varphi_r(x-y) = r^{-d-1}\nabla\varphi_1((x-y)/r)$ and taking absolute values everywhere, we get

$$|\nabla u(x)| \leq r^{-d-1} \int_{B_r(x)} \left| \nabla\varphi_1\left(\frac{x-y}{r}\right) \right| |u(y)| \, \mathrm{d}y \leq Cr^{-1} \max_{B_r(x)} |u|,$$

where $C = |B_1| \max |\nabla\varphi_1|$. A similar computation applies to higher order derivatives as well. □

The last proposition allows us to prove the Liouville theorem for harmonic functions.

Theorem 4.3.6 (Liouville theorem)**.** *Let $u : \mathbb{R}^d \to \mathbb{R}$ be a bounded function so that $\Delta u = 0$ in the whole $\mathbb{R}^d$. Then u must be constant.*

Proof. For any $x \in \mathbb{R}^d$ and $r > 0$ large, we apply Proposition 4.3.5 and obtain

$$|\nabla u(x)| \leq \frac{C}{r} \sup_{\mathbb{R}^d} |u|.$$

Taking $r \to +\infty$, we deduce that $\nabla u(x) = 0$ for any $x \in \mathbb{R}^d$. Then u must be constant. □

4.4 The fundamental solution.

The fundamental solution Φ of the Laplace equation is a radial function such that $\Delta\Phi = 0$ everywhere in $\mathbb{R}^d$ except at the origin. It is insightful to think that Φ is the function such that $\Delta\Phi = \delta_0$. Recall that the Dirac delta δ_0 is not a function, but a formal object defined in terms of how it integrates.

Let us now compute the fundamental solution for each value of the dimension d.

Let us write Φ in polar coordinates as $\Phi(r, \theta)$ where $\theta \in \partial B_1$. Since we are looking for a radially symmetric function, Φ should only depend on r. In this case, we have

$$\Delta\Phi = \partial_{rr}\Phi + \frac{d-1}{r}\Phi_r.$$

We are looking for a nontrivial solution Φ of this equation. We observe that this is equivalent to

$$\partial_r \left[r^{d-1} \partial_r \Phi \right] = 0.$$

Thus, we get $r^{d-1}\partial_r\Phi = c$ for some constant c. And we obtain the formula

$$\Phi(x) := \begin{cases} c \log |x|, & \text{if } d = 2, \\ -\frac{c}{|x|^{d-2}}, & \text{if } d > 2. \end{cases}.$$

So far, there is no reason to choose any constant c over another. We will pick c so that $|\nabla\Phi(x)|$ is equal to the inverse of the surface area of the sphere ∂B_1 when $|x| = 1$. This choice is made with the only purpose of making the following important result true.

Theorem 4.4.1. *Let $f : \mathbb{R}^d \to \mathbb{R}$ be C^1 and compactly supported. Then the function $u = \Phi * f$ solves $\Delta u = f$ in $\mathbb{R}^d$.*

Proof. Recall that when we take a derivative of a convolution, we can place it in either one of the two functions inside. In this case, we will spread the second derivatives in the Laplacian into both functions Φ and f. We have

$$\Delta u = \sum \partial_{x_i}^2 [\Phi * f] = \sum (\partial_{x_i}\Phi) * (\partial_{x_i} f) = \nabla\Phi * \nabla f.$$

This means

$$\Delta u(x) = \int_{\mathbb{R}^d} \nabla\Phi(y) \cdot \nabla f(x-y) \, \mathrm{d}y.$$

For any $\varepsilon > 0$, we split the domain of integration.

$$\Delta u(x) = \int_{B_\varepsilon} \nabla\Phi(y) \cdot \nabla f(x-y) \, \mathrm{d}y + \int_{\mathbb{R}^d \setminus B_\varepsilon} \nabla\Phi(y) \cdot \nabla f(x-y) \, \mathrm{d}y.$$

We integrate by parts in the second integral.

$$\Delta u(x) = \int_{B_\varepsilon} \nabla\Phi(y) \cdot \nabla f(x-y) \, \mathrm{d}y - \int_{\partial B_\varepsilon} \partial_\nu \Phi(y) f(x-y) \, \mathrm{d}S + \int_{\mathbb{R}^d \setminus B_\varepsilon} \Delta\Phi(y) f(x-y) \, \mathrm{d}y.$$

The third term is equal to zero since $\Delta\Phi = 0$ away from the origin. For the first two terms, we use the formula for Φ.

$$\Delta u(x) = \int_{B_\varepsilon} \tilde{C} \frac{y}{|y|^d} \cdot \nabla f(x-y) \,\mathrm{d}y + \frac{1}{|\partial B_\varepsilon|} \int_{\partial B_\varepsilon} f(x-y) \,\mathrm{d}S$$

In the last term, we used the choice of the constant C so that $\partial_\nu \Phi$ is constant on ∂B_ε and equal to $\varepsilon^{-d+1}/|\partial B_1| = 1/|\partial B_\varepsilon|$.

We now want to take the limit as $\varepsilon \to 0$. Estimating the absolute value of the first term we get

$$\left| \int_{B_\varepsilon} \tilde{C} \frac{y}{|y|^d} \cdot \nabla f(x-y) \,\mathrm{d}y \right| \leq C\varepsilon \max |\nabla f| \to 0 \quad \text{as } \varepsilon \to 0.$$

The second term is the average of f on the sphere $\partial B_\varepsilon(x)$. It converges to $f(x)$ as $\varepsilon \to 0$ because of the continuity of f. □

4.4.1 The Poisson kernel and Green function

Given any domain $\Omega \in \mathbb{R}^d$, we are looking for a formula to solve the Laplace equation

$$\begin{aligned} \Delta u &= f \text{ in } \Omega, \\ u &= g \text{ on } \partial\Omega. \end{aligned} \tag{4.5}$$

Any such formula has to depend linearly on f and g. Thus, it is natural to expect it to have the form

$$u(x) = \int_\Omega f(y) G(x,y) \,\mathrm{d}y + \int_{\partial\Omega} g(y) P(x,y) \,\mathrm{d}S(y). \tag{4.6}$$

Here, the functions G and P are special kernels depending on the domain Ω. We call them *Green function* and *Poisson kernel* respectively. It is possible to prove that they always exist and the formula above holds as long as the domain Ω has a smooth boundary. There is a relatively simple relation between them that we analyze below. We can compute them when Ω is the upper half space or a ball. For general domains Ω, it is in general impossible to obtain an explicit formula for them.

Let us begin by focusing on the first term of (4.5). That is, let us consider an equation where $g = 0$.

$$\begin{aligned} \Delta u_1 &= f \text{ in } \Omega, \\ u_1 &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{4.7}$$

A first attempt to solve the equation (4.7) may be to use the fundamental solution Φ of the Laplace equation. If we consider the expression

$$v_1(x) = \int_\Omega \Phi(x-y) f(y) \,\mathrm{d}y,$$

we can repeat our previous computation and see that $\Delta v_1 = f$. However, there is no reason for this function v_1 to vanish on the boundary $\partial\Omega$. Since u_1 and v_1 have the same Laplacian in Ω, the difference $v_1 - u_1$ is a harmonic function in Ω, with the same boundary values as v_1. We will compute u_1 by subtracting an appropriate harmonic function to v_1. Suppose that for each fixed value of $y \in \Omega$, we can find a harmonic function h_y such that

$$\begin{aligned} \Delta h_y(x) &= 0 \text{ for } x \in \Omega, \\ h_y(x) &= \Phi(x-y) \text{ for } x \in \partial\Omega. \end{aligned} \tag{4.8}$$

In terms of this auxiliary function, we can define $G(x,y) = \Phi(x-y) - h_y(x)$. We claim that

$$u_1(x) = \int_\Omega G(x,y) f(y) \,\mathrm{d}y.$$

Indeed, we have

$$\begin{aligned}\int_\Omega G(x,y)f(y)\,\mathrm{d}y &= \int_\Omega \left(\Phi(x-y) - h_y(x)\right) f(y)\,\mathrm{d}y,\\ &= v_1(x) - \int_\Omega h_y(x) f(y)\,\mathrm{d}y.\end{aligned}$$

The second term is a harmonic function in Ω with the same boundary values as v_1, which is what we were looking for.

The procedure described above allows us to obtain the Green function $G(x,y)$ provided that we can solve the auxiliary problem (4.8). Later, in Section 4.6, we will show that this equation is solvable in a fair amount of generality. It can be proven, although it would require a substantial amount of work, that the Green function $G(x,y)$ that we constructed is in general smooth up to the boundary, provided that the boundary $\partial\Omega$ is smooth. We will compute the Green function explicitly in the case that Ω is a ball, or the upper half space. For the following results, let us consider any scenario where we have a smooth Green function $G(x,y)$.

Proposition 4.4.2. *Let $G(x,y)$ be the Green function of the Laplace equation in Ω. Then G is a symmetric kernel, i.e. $G(x,y) = G(y,x)$ for all $x,y \in \Omega$.*

Proof. Let f_1 and f_2 be two arbitrary functions in Ω. Let us solve the two problems

$$\begin{aligned}\Delta u_1 = f_1 &\text{ and } \Delta u_2 = f_2 \text{ in } \Omega,\\ u_1 = u_2 &= 0 \text{ on } \partial\Omega.\end{aligned}$$

According to Green's identity, we have

$$\int_\Omega u_1 \Delta u_2 - u_2 \Delta u_1\,\mathrm{d}y = \int_{\partial\Omega} (u_1 \nabla u_2 - u_2 \nabla u_1)\cdot n\,\mathrm{d}S = 0.$$

Thus,

$$\begin{aligned}0 &= \int_\Omega u_1 f_2 - u_2 f_1\,\mathrm{d}y,\\ &= \int_\Omega \left(\int_\Omega G(x,y) f_1(y)\,\mathrm{d}y\right) f_2(x) - \left(\int_\Omega G(y,x) f_2(y)\,\mathrm{d}y\right) f_1(x)\,\mathrm{d}x,\end{aligned}$$

Interchanging the order of integration and swapping the variables in the second term,

$$= \iint_{\Omega\times\Omega} [G(x,y) - G(y,x)] f_1(y) f_2(x)\,\mathrm{d}x\,\mathrm{d}y,$$

The only way for this integral to be zero for all functions f_1 and f_2 is if $G(x,y) = G(y,x)$ for all $x,y \in \Omega$. This finishes the proof. □

A way to think about the Green function is that it is a solution to the Laplace equation with a Dirac delta source at y. The Green function $G(x,y)$ is the solution to the equation $\Delta_x G(x,y) = \delta_y(x)$ with $G(x,y) = 0$ when $x \in \partial\Omega$. Since $G(x,y)$ is symmetric with respect to x and y, it is also the solution to $\Delta_y G(x,y) = \delta_x(y)$ with $G(x,y)$ whenever $y \in \partial\Omega$.

We want to understand both terms in formula (4.5). Let us proceed with the following (in)formal computation to obtain an expression for the Poisson kernel $P(x,y)$ in terms of the Green function $G(x,y)$.

$$\begin{aligned}u(x) &= \int_\Omega u(y)\delta_x(y)\,\mathrm{d}y,\\ &= \int_\Omega u(y)\Delta_y G(x,y)\,\mathrm{d}y,\end{aligned}$$

Integrating by parts twice, and writing ν for the exterior unit vector normal to $\partial\Omega$,

$$
\begin{aligned}
&= -\int_\Omega \nabla u(y)\cdot\nabla_y G(x,y)\,\mathrm{d}y + \int_{\partial\Omega} u(y)\nabla_y G(x,y)\cdot\nu\,\mathrm{d}S,\\
&= \int_\Omega \Delta u(y) G(x,y)\,\mathrm{d}y + \int_{\partial\Omega} u(y)(\nabla_y G(x,y)\cdot\nu) - (\nabla u(y)\cdot\nu)G(x,y)\,\mathrm{d}S,\\
&= \int_\Omega f(y)G(x,y)\,\mathrm{d}y + \int_{\partial\Omega} g(y)(\nabla_y G(x,y)\cdot\nu)\,\mathrm{d}S.
\end{aligned}
$$

In the last line, we used that $G(x,y)=0$ if $y\in\partial\Omega$. We also used the fact that $u(y)=g(y)$ on $\partial\Omega$. We effectively derived the formula (4.5) with $P(x,y)=\nabla_y G(x,y)\cdot\nu$. Thus, the Poisson kernel is the normal derivative of the Green function on the boundary $\partial\Omega$.

In the following sections, we will compute explicitly the Green function for the simple cases of the ball and the upper half space.

The Green function of the upper half space

Let us consider the domain Ω that consists in the upper half space

$$\Omega = \{x=(x_1,x_2,\dots,x_d)\in\mathbb{R}^d : x_d>0\}.$$

This is an unbounded domain, the formula (4.6) will hold and provides us with a solution to (4.5) only if the functions u, f and g have sufficient decay at infinity.

The solution of the problem (4.8) is obtained by reflecting the fundamental solution across the plane $x_d=0$. Indeed, for any $x=(x_1,\dots,x_d)\in\mathbb{R}^d$, let us consider its *conjugate* $\bar{x}=(x_1,\dots,x_{d-1},-x_d)$. Then the solution to (4.8) is given by

$$\phi_x(y)=\Phi(\bar{x}-y).$$

The function $\phi_x(y)$ clearly coincides with $\Phi(x-y)$ whenever $y_d=0$, since in that case $|x-y|=|\bar{x}-y|$ Moreover, for any $x\in\Omega$, the function ϕ_x is harmonic away from $y=\bar{x}$. But $\bar{x}\notin\Omega$, so ϕ_x is the solution to (4.8). Therefore, in this case

$$G(x,y)=\Phi(x-y)-\Phi(\bar{x}-y).$$

Once we obtained the Green function, we compute the Poisson kernel $P(x,y)$ from the formula $P(x,y)=\nu\cdot\nabla G(x,y)$. In this case $\nu=(0,\dots,0,-1)$, thus for $x\in\Omega$ and $y_d=0$,

$$P(x,y)=\partial_d\Phi(x-y)-\partial_d\Phi(\bar{x}-y)=2\partial_d\Phi(x-y)=C_d\frac{x_d}{|x-y|^d},$$

for a constant C_d depending on dimension only.

The Green function of the unit ball

Let us now analyze the domain $\Omega=B_1$ in $\mathbb{R}^d$. Given any point $x\in B_1$, we also consider the point $\bar{x}$ obtained by *inversion* through the sphere. It is, by definition,

$$\bar{x}:=\frac{x}{|x|^2}.$$

For any point $x\in\mathbb{R}^d$, The point x and $\bar{x}$ are aligned and $|x||\bar{x}|=1$. In particular, inversion maps the interior of the sphere to its exterior, and keeps ∂B_1 fixed.

With this notation, we solve the problem (4.8) explicitly by taking

$$\phi_x(y)=\Phi\left(|x|(\bar{x}-y)\right).$$

It is clear that this function ϕ_x is harmonic away from $\bar{x}$. In particular, ϕ_x is harmonic in B_1 for any $x\in B_1$. Some justification is required to verify that $\Phi(x-y)=\phi_x(y)$ whenever $y\in\partial B_1$. This follows from the fact that $|x-y| = |x||\bar{x}-y|$. In order to verify this identity, for any $y \in \partial B_1$, let us square the right hand side of this equality and expand it

$$|x|^2|\bar{x}-y|^2 = |x|^2\left(|x|^{-2} - 2x\cdot y/|x|^2 + |y|^2\right),$$
$$= 1 - 2x\cdot y + |x|^2 = |y-x|^2.$$

The case $x = 0$ is special. When $x = 0$, there is a division by zero in our expression for $\bar{x}$. Since Φ is a radially symmetric function, we observe that $\Phi(x-y)$ is constant on ∂B_1 when $x = 0$. Thus, the function $\phi_0(y)$ is simply a constant function. In fact, one can verify that $\Phi(|x|(\bar{x}-y))$ converges to this constant as $x \to 0$.

We therefore get the following formula for the Green function in the unit ball

$$G(x,y) = \Phi(x-y) - \Phi(|x|(\bar{x}-y)).$$

The Poisson kernel is computable using the expression $P(x,y) = \nu \cdot \nabla G(x,y)$. A similar computation works appropriately scaled in balls of any radius and centered anywhere in $\mathbb{R}^d$.

The explicit construction of the Green and Poisson kernels provides us with an effective way to solve the equation (4.5) in a ball. We state it as the next proposition.

Proposition 4.4.3 (Solvability in the ball)**.** *For any functions $f : \partial B_r \to \mathbb{R}$ continuous and $g : B_r \to \mathbb{R}$ differentiable, there exists one function u which is continuous in $\overline{B}_r$, C^2 in B_r and solves*

$$u = f \text{ on } \partial B_r,$$
$$\Delta u = g \text{ in } B_r.$$

We will not compute of the explicit formula for the Poisson kernel of the ball. In order to fully justify Proposition 4.4.3, it would be necessary the verify a posteriori that the formula (4.6) effectively solves (4.5). The proof would be similar as to what we did for the Poisson kernel in the two dimensional case, when we solved the Laplace equation in a disc using the method of separation of variables. We omit this computation as well.

4.5 Sub/super-harmonic functions

Definition 4.5.1. *We say a continuous function $u : \Omega \to \mathbb{R}$ is subharmonic, if for any $x \in \Omega$ there is an $r_0 > 0$ such that for all $r < r_0$, we have*

$$u(x) \le \frac{1}{|B_r|}\int_{B_r(x)} u(y)\,\mathrm{d}y. \tag{4.9}$$

We say u is superharmonic if the opposite inequality is verified, that is

$$u(x) \ge \frac{1}{|B_r|}\int_{B_r(x)} u(y)\,\mathrm{d}y.$$

Remark 4.5.2. Following the proof of the mean value property, we can verify that a C^2 function u is subharmonic if and only if $\Delta u \ge 0$ in Ω. Likewise, a C^2 function u is superharmonic if and only if $\Delta u \le 0$ in Ω. The advantage of our definition above is that it does not require the function u to be differentiable.

Lemma 4.5.3. *The maximum of two subharmonic functions is also subharmonic.*

Proof. The maximum of two continuous functions is always continuous. We have to verify that the inequality (4.9) is satisfied at every point.

Let $u, v : \Omega \to \mathbb{R}$ be two subharmonic functions. Let $x \in \Omega$ be any point. We know that $\max(u(x), v(x))$ is equal to either $u(x)$ or $v(x)$. Let us say that it is equal to $u(x)$. Then, using the subharmonicity of u, we deduce

$$\max(u(x),v(x)) = u(x) \le \frac{1}{|B_r|}\int_{B_r(x)} u(y)\,\mathrm{d}y \le \frac{1}{|B_r|}\int_{B_r(x)} \max(u(y),v(y))\,\mathrm{d}y.$$

Thus, we have verified the inequality (4.9) for $\max u, v$ and we are done. ☐

Lemma 4.5.4. *Assume $u : \Omega \to \mathbb{R}$ is subharmonic and $v : \Omega \to \mathbb{R}$ is superharmonic. Then $u - v$ is subharmonic.*

Proof. The difference of two continuous functions is also continuous. We must verify the inequality (4.9).

Since u is subharmonic and v is superharmonic, for any $x \in \Omega$ and for r sufficiently small, we have

$$u(x) \leq \frac{1}{|B_r|} \int_{B_r(x)} u(y) \,\mathrm{d}y \quad \text{and} \quad v(x) \geq \frac{1}{|B_r|} \int_{B_r(x)} v(y) \,\mathrm{d}y.$$

Subtracting the two inequalities, we get

$$u(x) - v(x) \leq \frac{1}{|B_r|} \int_{B_r(x)} u(y) - v(y) \,\mathrm{d}y.$$

□

Lemma 4.5.5 (The strong maximum principle for subharmonic functions)**.** *Let $u : \Omega \to \mathbb{R}$ be subharmonic. Assume Ω is connected and open and u achieves its maximum in an interior point in Ω. Then u is constant in Ω.*

Proof. Let $M = \max_\Omega u$. We know that there exists an $x_0 \in \Omega$ such that $u(x_0) = M$.

Let $A = \{x \in \Omega : u(x) = M\}$. We know $x_0 \in A$, so A is non empty. Since u is continuous, then also A is relatively closed (as a subset of Ω). If we prove that A is relatively open, then we would have a non empty subset of a connected set which is at the same time open and closed, and this implies $A = \Omega$ because of the definition of connectedness.

Let $x \in A$. Since u is subharmonic, for r sufficiently small we have

$$M = u(x) \leq \frac{1}{|B_r|} \int_{B_r(x)} u(y) \,\mathrm{d}y.$$

We observe that $u(y) \leq M$ for all $y \in B_r(x) \subset \Omega$. Therefore, the right hand side is less than or equal to M. We must have equality, but equality is only achieved if $u(y) \equiv M$ in $B_r(x)$. Thus, we deduced that $B_r(x) \subset A$. This means that A is an open set and we finish the proof. □

Corollary 4.5.6. *Let Ω be a bounded open set. Let $u : \overline{\Omega} \to \mathbb{R}$ be continuous in $\overline{\Omega}$ and subharmonic in Ω. Then*

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

Proof. Assume the equality is not true. Then we would have a point $x_0 \in \Omega$ such that $u(x_0) = \max_\Omega u$ and $u(x_0)$ is strictly larger than the value of u at any point on the boundary $\partial\Omega$.

Let $R \subset \Omega$ be the connected component of Ω such that $x_0 \in R$ (i.e. R is the union of all connected subsets of Ω which contain x_0). We can verify that R reaches the boundary of Ω, i.e. $\overline{R} \cap \partial\Omega \neq \emptyset$ (it is a stardard property of connected sets, do it as an exercise).

Since u is continuous on $\overline{\Omega}$, we get that there must be a point on the boundary $\partial\Omega$ where u equals $u(x_0)$, which contradicts our hypothesis. □

Corollary 4.5.7 (The comparison principle)**.** *Assume Ω is a bounded open set. Let $u, v : \overline{\Omega} \to \mathbb{R}$ be continuous in $\overline{\Omega}$. Assume u is subharmonic in Ω and v is superharmonic in Ω. If $u \leq v$ on $\partial\Omega$ then $u \leq v$ in the whole set Ω.*

Proof. From Lemma 4.5.4, $u - v$ is subharmonic. We finish the proof applying Corollary 4.5.6 to $u - v$. □

4.6 Perron's method

The purpose of this section is to construct a solution to the Dirichlet problem

$$\begin{aligned} \Delta u &= 0 \text{ in } \Omega, \\ u &= f \text{ on } \partial\Omega. \end{aligned} \tag{4.10}$$

We assume Ω is a bounded open set with a smooth boundary and f is a continuous function on $\partial\Omega$.

Because of the comparison principle of Corollary 4.5.7, we see that any subharmonic function whose boundary values are less than f will be less than our solution u. Because of this, it makes sense to define u as

$$u(x) = \sup\left\{v(x) : v \text{ is subharmonic in } \Omega \text{ and } v \leq f \text{ on } \partial\Omega\right\}. \tag{4.11}$$

We are left to verify that this function u solves (4.10).

We note first that the set of functions v is non empty. Indeed, since f is continuous and Ω is bounded, $f \geq -M$ for some large M. Then the function $v \equiv -M$ is one admissible function in the set. Moreover, this implies the bound from below $u \geq v = -M$.

Note that any function v which takes values less than $-M$ is irrelevant for the supremum. Indeed, because of Lemma 4.5.3, $\max(v, -M)$ would also be an admissible function which is larger or equal than v everywhere in Ω. This means that we can look at the supremum among a set of uniformly bounded functions. Indeed, because of Corollary 4.5.7 (the comparison principle) all subharmonic functions v also satisfy $v \leq \max f$.

We now explain another procedure that enlarges an admissible function v. Given any such function v and a ball $B_\rho(x_0) \subset \Omega$, there is a unique function w such that

$$\begin{aligned} w &= v \text{ in } \overline{\Omega} \setminus B_\rho(x_0), \\ \Delta w &= 0 \text{ in } B_\rho(x_0). \end{aligned} \tag{4.12}$$

The function w inside $B_\rho(x_0)$ is obtained solving the Dirichlet problem with boundary conditions on $\partial B_\rho(x_0)$ given by the values of v (see Proposition 4.4.3). Note that the function w depends on v and the choice of ball $B_\rho(x_0)$.

Lemma 4.6.1. *The function w is continuous, subharmonic in Ω and $w \geq v$ in Ω.*

Proof. We know that v is continuous, and then w is continuous outside of $B_\rho(x_0)$. The continuity of w on $\overline{B}_\rho(x_0)$ follows from the continuity of the solution to the Dirichlet problem in the ball as in Proposition 4.4.3. We also get $w \geq v$ in $B_\rho(x_0)$ because of Corollary 4.5.7. We are left to prove that w is subharmonic in Ω.

Let x be any point in Ω. If $x \notin B_\rho(x_0)$, then $w(x) = v(x)$. Since $w \geq v$ and v is subharmonic, it follows that for r small,

$$w(x) = v(x) \leq \frac{1}{|B_r|}\int_{B_r(x)} v(y) \, dy \leq \frac{1}{|B_r|}\int_{B_r(x)} w(y) \, dy.$$

If $x \in B_\rho(x_0)$, we use that w is harmonic in $B_\rho(x_0)$. Therefore, for r small,

$$w(x) = \frac{1}{|B_r|}\int_{B_r(x)} w(y) \, dy.$$

In any case, we proved that w is subharmonic. □

Lemma 4.6.2. *The function u defined by Perron's method is harmonic in Ω.*

Proof. Let $x_0 \in \Omega$. Let v_n be a sequence of admissible subharmonic functions such that

$$u(x_0) = \lim_{n\to\infty} v_n(x_0).$$

We explained above that we can assume that v_n are uniformly bounded. Let $\rho > 0$ be sufficiently small so that $B_\rho(x_0) \subset \Omega$. We define w_n to the the enlarged subharmonic function corresponding to v_n as in (4.12).

Because of Theorem 4.3.4, we have that

$$\max_{B_{\rho/2}(x_0)} |\nabla w_n| \leq \frac{C}{\rho} \max_{B_\rho(x_0)} |w_n|.$$

Therefore, the sequence of functions w_n is equicontinuous. By the theorem of Arzela-Ascoli, there exists a uniformly convergent subsequence. We call $w : B_{\rho/2}(x_0) \to \mathbb{R}$ its uniform limit.

For every ball $B_r(x) \subset B_{\rho/2}(x_0)$, we have that

$$w_n(x) = \frac{1}{|B_r|} \int_{B_r(x_0)} w_n(y) \, \mathrm{d}y.$$

Using uniform convergence and passing to the limit as $n \to \infty$, we get

$$w(x) = \frac{1}{|B_r|} \int_{B_r(x_0)} w(y) \, \mathrm{d}y.$$

Since this equality holds at every point $x \in B_{\rho/2}(x_0)$, then w is harmonic in $x \in B_{\rho/2}(x_0)$.

Our purpose now is to show that $u = w$ in $B_{\rho/2}(x_0)$.

Let x_1 be any point in $B_{\rho/2}(x_0)$. As before, we can find a sequence of admissible subsolutions $\tilde{v}_n$ so that $\lim_{n\to\infty} \tilde{v}_n(x_1) = u(x_1)$. From Lemma 4.5.3, we know that $\max(v_n, \tilde{v}_n)$ is also an admissible subsolution. We construct, as before, the harmonic replacement of $\max(v_n, \tilde{v}_n)$ in $B_\rho(x_0)$ and call it $\tilde{w}_n$.

$$\begin{aligned} \tilde{w}_n &= \max(v_n, \tilde{v}_n) \text{ in } \overline{\Omega} \setminus B_\rho(x_0), \\ \Delta \tilde{w}_n &= 0 \text{ in } B_\rho(x_0). \end{aligned} \tag{4.13}$$

Since $\max(v_n, \tilde{v}_n) \geq v_n$, then also $\tilde{w}_n \geq w_n$ because of Corollary 4.5.7.

As before, we extract a subsequence so that $\tilde{w}_n$ converges uniformly in $B_{\rho/2}(x_0)$ to a harmonic function $\tilde{w}$. We will have $\tilde{w} \geq w$ in $B_{\rho/2}(x_0)$, but since $\tilde{w}(x_0) = w(x_0) = u(x_0)$, then we conclude that $\tilde{w} \equiv w$ in $B_{\rho/2}(x_0)$ because of the strong maximum principle.

This means that $w(x_1) = \tilde{w}(x_1) = u(x_1)$. Since x_1 was an arbitrary point in $B_{\rho/2}(x_0)$, we have that $w \equiv u$ in $B_{\rho/2}(x_0)$. Thus, u is C^∞ and $\Delta u = 0$. □

Note that all we have used to deduce that the function u given by Perron's method is harmonic in Ω is that Ω is an open set. An extra smoothness condition of $\partial\Omega$ is a requirement in order to prove that indeed $u = f$ on $\partial\Omega$.

Definition 4.6.3. *We say that a function $B : \overline{\Omega} \to \mathbb{R}$ is a barrier at $x_0 \in \partial\Omega$ if*

- $B(x_0) = 0$,
- $B(x) > 0$ *for all* $x \in \partial\Omega \setminus \{x_0\}$,
- B *is superharmonic in* Ω.
- B *is continuous in* $\overline{\Omega}$.

Lemma 4.6.4. *Assume that Ω is bounded, f is continuous and there exists some barrier at every point $x_0 \in \partial\Omega$. Then the function u from Perron's method is continuous and equals f on $\partial\Omega$.*

Proof. Let $\varepsilon > 0$ and $x_0 \in \partial\Omega$ be arbitrary. Let B be the barrier function at x_0.

Since f is a continuous on $\partial\Omega$, then there exists a $\delta_1 > 0$ such that $|f(x) - f(x_0)| \leq \varepsilon/2$ if $|x - x_0| < \delta_1$.

Let $m = \min\{B(x) : x \in \partial\Omega \setminus B_{\delta_1}(x_0)\} > 0$ and $M = 2 \max |f|/m$.

We verify that u is less than or equal to

$$U(x) = f(x_0) + \frac{\varepsilon}{2} + M\, B(x)$$

on $\partial\Omega$. Since u is harmonic and U is superharmonic, the inequality is extended to the interior of Ω using Corollary 4.5.7.

Since B is continuous and $B(x_0) = 0$, for $\delta > 0$ sufficiently small, $B(x) < \varepsilon/(2M)$ if $|x - x_0| < \delta$. Thus, for $|x - x_0| < \delta$ we have

$$u(x) - f(x_0) \leq U(x) - f(x_0) \leq \varepsilon.$$

Analogously, we define

$$v(x) = f(x_0) - \frac{\varepsilon}{2} - M\,B(x).$$

This is an admissible subharmonic function. Then $u \geq v$ in Ω and

$$u(x) - f(x_0) \geq v(x) - f(x_0) \geq -\varepsilon$$

for $|x - x_0| < \delta$.

Therefore, we proved by definition that

$$\lim_{x \to x_0} u(x) = f(x_0),$$

and x_0 is an arbitrary point on $\partial\Omega$. □

The previous lemma tells us that Perron's method will succeed in reproducing the boundary conditions provided that barrier functions exist. We need another result which provides us with some conditions that guarantee the existence of these special functions. That is the purpose of the next lemma.

Lemma 4.6.5. *Let $x_0 \in \Omega$ be a point such that there exists a ball $B_r(x_1)$ such that $B_r(x_1) \cap \Omega = \emptyset$ and $\partial B_r(x_1) \cap \partial\Omega = \{x_0\}$. Then there exist a barrier at x_0.*

Proof. The proof of the lemma is the explicit construction

$$B(x) = \begin{cases} \log|x - x_1| - \log r, & \text{if } n = 2, \\ |x - x_1|^{2-n} - r^{2-n}, & \text{if } n > 2. \end{cases}$$

which satisfies the assumptions of Definition 4.6.3. □

We say that $\partial\Omega$ at a point x_0 as in Lemma 4.6.5 satisfies the exterior sphere condition.

Combining all the results above, we obtain the following final theorem.

Theorem 4.6.6. *Assume Ω is bounded, $\partial\Omega$ satisfies the exterior sphere condition at all its points, and $f : \partial\Omega \to \mathbb{R}$ is continuous. Then the equation* (4.10) *has a unique solution u, which is given by the Perron's method* (4.11)*.*

4.7 Exercises

Exercise 4.1.

(a) Let $u \in C^2(\overline{\Omega})$ be a solution of

$$\begin{aligned} \Delta u &= g \quad \text{in } \Omega, \\ u &= f \quad \text{on } \partial\Omega. \end{aligned}$$

Prove that for any other function $v \in C^1(\overline{\Omega})$ so that $v = f$ on $\partial\Omega$, we have

$$\int_\Omega \frac{|\nabla u|^2}{2} + gu \,\mathrm{d}x \leq \int_\Omega \frac{|\nabla v|^2}{2} + gv \,\mathrm{d}x.$$

(b) *Let* $u \in C^2(\overline{\Omega})$ *be a solution of*

$$\Delta u - u^3 = 0 \quad \text{in } \Omega,$$
$$u = f \quad \text{on } \partial\Omega.$$

Prove that for any other function $v \in C^1(\overline{\Omega})$ *so that* $v = f$ *on* $\partial\Omega$, *we have*

$$\int_\Omega \frac{|\nabla u|^2}{2} + \frac{u^4}{4}\, \mathrm{d}x \le \int_\Omega \frac{|\nabla v|^2}{2} + \frac{v^4}{4}\, \mathrm{d}x.$$

Exercise 4.2. *Let* u *be a harmonic function in the unit disk in* $\mathbb{R}^2$. *Consider the function*

$$v(x, y) = u(e^{-y}\cos x, e^{-y}\sin x).$$

Prove that v *is harmonic in the upper half space:* $\{(x, y) : y > 0\}$.

Exercise 4.3. *Let* $\Omega \subset \mathbb{R}^d$ *be a bounded open set,* $c : \Omega \to \mathbb{R}$ *be continuous and non negative, and* $a_{ij} : \Omega \to \mathbb{R}$ *be continuous for* $i, j = 1, \ldots, d$, *so that the matrix* $a_{ij}(x)$ *is (strictly) positive definite for all* $x \in \Omega$. *Let* u *be a solution to the following* elliptic *equation*

$$\sum_{i,j=1,\ldots,d} a_{ij}(x)\partial_{ij}u(x) - c(x)u(x) \ge 0 \quad \text{in } \Omega.$$

(a) *Prove the following maximum principle: if* $u \le 0$ *on* $\partial\Omega$, *then* $u \le 0$ *in* Ω.

(b) *Deduce that the following Dirichlet problem has at most one solution*

$$\sum_{i,j=1,\ldots,d} a_{ij}(x)\partial_{ij}u(x) - c(x)u(x) = 0 \quad \text{in } \Omega,$$
$$u = f \quad \text{on } \partial\Omega.$$

(c) *Give an example to show that uniqueness of solutions may not hold if* $c < 0$.

Hint. The following fact from linear algebra is used for the previous question: if A and B are two positive semi-definite matrices, then $\sum_{ij} A_{ij}B_{ij} \ge 0$. Can you verify it?

Exercise 4.4. *Let* Ω *be a bounded open set in* $\mathbb{R}^d$ *and* u *solve the equation*

$$\Delta u = -1 \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega.$$

Prove that $u \le \frac{1}{2d}(\operatorname{diam}\Omega)^2$.

Here $\operatorname{diam}\Omega = \sup\{|x - y| : x, y \in \Omega\}$ *stands for the diameter of the set* Ω.

Exercise∗ 4.5. *For one of the following two equations the maximum principle holds and for the other one it does not. Prove it for the one which holds, and give a counterexample for the other.*

For both equations we look for a C^2 *function* $u : \overline{B}_1 \to \mathbb{R}$. *The set* B_1 *is the unit disc in two dimensions*

$$B_1 = \{(x, y) = x^2 + y^2 < 1\}.$$

Equation 1.

$$y^2u_{xx} + x^2u_{yy} - 2xyu_{xy} = 0 \text{ in } B_1.$$

Equation 2.

$$y^2u_{xx} + x^2u_{yy} - 2xyu_{xy} - xu_x - yu_y = 0 \text{ in } B_1.$$

Hint. The equations have a fairly simple form in polar coordinates.

Hint. For the proof in the positive case, you want to repeat the argument of the proof of the weak maximum principle for the Laplace equation, adding $\varepsilon|x|$ to the solution and look for the point where its maximum is achieved. Note that the equation gives no information at the point $(0, 0)$.

Exercise 4.6. *Let $u : \Omega \to \mathbb{R}$ be a C^2 function so that $\Delta u \geq 0$ everywhere in Ω. Prove that for any ball $B_r(x) \subset \Omega$, we have*

$$u(x) \leq \frac{1}{|\partial B_r|} \int_{\partial B_r(x)} u(y) \, \mathrm{d}S(y).$$

Exercise 4.7. *Let u be a harmonic function in the annulus $B_{R_2} \setminus B_{R_1}$. Let $F : [R_1, R_2] \to \mathbb{R}$ be the function*

$$F(r) = \frac{1}{|\partial B_r|} \int_{\partial B_r} u(y) \, \mathrm{d}S(y),$$

Prove that $r^{d-1}F'(r)$ is constant, but F may not be.

Exercise 4.8. *Let u be harmonic in $\mathbb{R}^d \setminus B_1$. Prove that there exist two constants c_0 and c_1 such that*

$$\frac{1}{|\partial B_r|} \int_{\partial B_r} u \, \mathrm{d}S = c_0 + c_1 \Phi(re).$$

Here Φ is the fundamental solution in $\mathbb{R}^d$ and e is an arbitrary unit vector.

Exercise 4.9. *Let $u : B_1 \to \mathbb{R}$ be harmonic and non negative. Prove that there exists a constant C depending on dimension only so that for any two points $x, y \in B_{1/4}$,*

$$u(x) \leq C u(y).$$

Hint. Use the mean value property in two balls centered in x and y so that one contains the other. This property of harmonic functions is called the *Harnack inequality*.

Exercise 4.10 (Harnack inequality)**.** *Let u be a nonnegative harmonic function in an open set Ω. Let $K \subset \Omega$ be compact and connected. Prove that there is a constant C depending only on the sets K and Ω only so that*

$$\max_K u \leq C \min_K u.$$

Hint. Cover the set K with small balls contained in Ω and use the previous question.

Exercise 4.11. *Let $u : B_r \to \mathbb{R}$, with $\Delta u = 0$ in B_r. Here B_r is the ball of radius r, centered at 0, in $\mathbb{R}^n$. Prove that*

$$|\nabla u(0)| \leq \frac{C}{r^{n+1}} \int_{B_r} |u(x)| \, \mathrm{d}x.$$

Let k be a positive integer. For what value of q (depending on k and n) is there a constant C for which the following inequality holds?

$$|D^{(k)} u(0)| \leq \frac{C}{r^q} \left(\int_{B_r} |u(x)|^2 \, \mathrm{d}x \right)^{1/2}.$$

Hint. In order to solve the last exercise, use the following version of the Cauchy-Schwarz inequality. For any functions f and g, we have

$$\int f(x) g(x) \, \mathrm{d}x \leq \left(\int |f(x)|^2 \, \mathrm{d}x \right)^{1/2} \left(\int |g(x)|^2 \, \mathrm{d}x \right)^{1/2}.$$

Exercise∗ 4.12. *Let $u : B_1 \to \mathbb{R}$ be a C^2 function that satisfies the equation*

$$\Delta u + |\nabla u|^2 = 0.$$

Prove that there is a constant C depending on dimension only so that $|\nabla u(0)| \leq C$.

Hint. The function $v(x) = \exp(u(x))$ is positive and harmonic.

Exercise 4.13. *Let $P : B_1 \times \partial B_1 \to \mathbb{R}$ be the Poisson kernel of the unit sphere.*

(a). Show that $P(0,y)$ *is constant as a function of* y *in* ∂B_1*.*

(b). Prove that for all $x \in B_1$*,*

$$\int_{\partial B_1} P(x,y)\, \mathrm{d}S(y) = 1.$$

(c). Prove that for all $y \in \partial B_1$*,*

$$\int_{\partial B_r} P(x,y)\, \mathrm{d}S(x) = r^{d-1}.$$

Hint. The Poisson kernel $P(x,y)$ is harmonic with respect to x for every $y \in \partial B_1$ fixed.

Exercise 4.14. *Let* P *and* G *be the Poisson kernel and the Green function of some arbitrary bounded domain* Ω*.*

(a) Prove that G *is nonpositive and* P *is nonnegative.*

(b) Prove that for any $x \in \Omega$*,*

$$\int_{\partial\Omega} P(x,y)\, \mathrm{d}S(y) = 1.$$

(c) Prove that for any $x \in \Omega$*,*

$$-\int_{\Omega} G(x,y)\, \mathrm{d}y \leq \frac{1}{2d}(\operatorname{diam}\Omega)^2.$$

Here $\operatorname{diam}\Omega = \sup\{|x-y| : x, y \in \Omega\}$ *stands for the diameter of the set* Ω*.*

Exercise∗ 4.15. *Let* $u : \overline{B}_1 \to \mathbb{R}$ *be harmonic and positive in* B_1*. Assume* $u(x_0) = 0$ *for some* $x_0 \in \partial B_1$*. Prove that* $u_\nu(x_0) < 0$*, where* ν *is the exterior unit normal vector to the boundary at* x_0*.*

Hint. if $d > 2$, prove that $u(x) \geq c(|x|^{2-d} - 1)$ in $B_1 \setminus B_{1/2}$ for c sufficiently small.
Note. This is known as *Hopf's lemma.*

Exercise 4.16. *Let* Ω *be an open set with a* C^2 *boundary and let* P *be its corresponding Poisson kernel. Prove that* $P(x,y) > 0$ *for any* $x \in \Omega$ *and* $y \in \partial\Omega$*.*

Hint. Since $\partial\Omega$ is C^2, for any $x \in \partial\Omega$ we can find a ball contained in Ω which is tangent to $\partial\Omega$ at x.

Exercise 4.17. *Let* u_n *be a sequence of harmonic functions in an open set* Ω*. Assume* u_n *converges uniformly to the function* u*.*

(a) Prove that u *is also harmonic.*

(b) Prove that ∇u_n *converges to* ∇u *uniformly over any compact subset of* Ω*.*

Exercise 4.18. *Let* $u : \mathbb{R}^n \to \mathbb{R}$ *be harmonic. That is* $\Delta u = 0$ *in the whole space* $\mathbb{R}^n$*. Assume also that* $u \geq 0$*. Prove that* u *is constant.*

Hint. Use the Harnack inequality.

Exercise 4.19. *Let* $u : \Omega \to \mathbb{R}$ *be a subharmonic function. Let* $B_r(x) \subset \Omega$*. Prove that*

$$u(x) \leq \frac{1}{|\partial B_r|} \int_{\partial B_r} u(y)\, \mathrm{d}S(y).$$

Consequently, we also have

$$u(x) \leq \frac{1}{|B_r|} \int_{B_r} u(y)\, \mathrm{d}y,$$

for any ball $B_r(x) \subset \Omega$ *(removing the condition that* r *has to be sufficiently small).*

Hint. Compare u with the function w solving the problem

$$\begin{aligned} \Delta w &= 0 \text{ in } B_r(x), \\ w &= u \text{ on } \partial B_r(x). \end{aligned}$$

Exercise 4.20. *Let u be a subharmonic function in Ω. Assume $B_R(x) \subset \Omega$ and let $F : [0, R] \to \mathbb{R}$ be given by*

$$F(r) = \int_{\partial B_r(x)} u(y) \,\mathrm{d}S.$$

Prove that F is monotone increasing.

Hint. For different values of ρ in $(0, R]$, consider the corresponding patch problems in $B_{\rho_i}(x)$ as in (4.12) (or the previous question).

Exercise 4.21. *Let $u : \Omega \to \mathbb{R}$ be subharmonic. Assume that there is a C^2 function φ for which $u - \varphi$ has a local maximum at the point $x_0 \in \Omega$. Prove that $\Delta\varphi(x_0) \geq 0$.*

Hint. Recall that $\Delta\varphi(x_0) = c \lim_{r\to 0} r^{-d-2} \int_{B_r(x_0)} (\varphi(y) - \varphi(x)) \,\mathrm{d}y$.

Exercise 4.22. *Let $u : \Omega \to \mathbb{R}$ be subharmonic and $\varphi : \mathbb{R}^n \to \mathbb{R}$ be smooth, nonnegative and supported in B_r. Let $\Omega_r = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > r\}$. Prove that the function*

$$u * \varphi(x) = \int_{\Omega} u(y)\varphi(x - y) \,\mathrm{d}y,$$

is subharmonic in Ω_r. Moreover, if $\varphi \in C^2$, then $u * \varphi \in C^2$ and $\Delta[u * \varphi] \geq 0$.

Exercise 4.23. *Let $u_i : \overline{\Omega} \to \mathbb{R}$ be an arbitrary sequence of sub-harmonic functions. Assuming that $v(x) :=$ $\sup\{u_i(x)\}$ is a continuous function on Ω, prove that v is sub-harmonic. Show an example in which the functions u_i are uniformly bounded and their supremum v is not continuous.*

Exercise 4.24.

(a) Show an example of a bounded subharmonic function in $\mathbb{R}^3$.

(b) Prove that any bounded subharmonic function in $\mathbb{R}^2$ is constant.

Hint. Part (b) is tricky. Start from a point x where $u(x) < \sup u$, then use the fundamental solution to build a convenient supersolution and get a contradiction.

Exercise$*$ 4.25. *Let Ω be an open set in $\mathbb{R}^d$ $(d \geq 2)$ and $u : \Omega \to \mathbb{R}$ be a harmonic function. Let*

$$\Omega' := \left\{ x : \frac{x}{|x|^2} \in \Omega \right\} \text{ and } v(x) = \frac{1}{|x|^{d-2}} u\left(\frac{x}{|x|^2}\right).$$

Prove that v is harmonic in Ω'.

Note. The function v is called the *Kelvin transform* of u.

Exercise 4.26. *Let u be a harmonic function in Ω. Assume $B_r \subset \Omega$. Prove the following identities.*

(a)

$$\int_{\partial B_r} u\, u_\nu \,\mathrm{d}S = \int_{B_r} |\nabla u|^2 \,\mathrm{d}x.$$

(b)

$$\int_{\partial B_r} |u_\tau|^2 - u_\nu^2 \,\mathrm{d}S = \frac{d-2}{r} \int_{B_r} |\nabla u|^2 \,\mathrm{d}x.$$

Here, u_ν denotes the directional derivative of u with respect to the exterior unit normal vector ν. We write u_τ to denote the component of ∇u that is tangential to ∂B_r. Thus, $|\nabla u|^2 = |u_\tau|^2 + u_\nu^2$.

Hint. For part (b), apply the divergence theorem to the vector field $|\nabla u|^2 x - 2(x\cdot\nabla u)\nabla u$.

Exercise∗ 4.27 (Hadamard's three-circle theorem for harmonic functions)**.** *Let u be a harmonic function in a domain $\Omega \subset \mathbb{R}^d$ that contains a ball B_R.*

(a) Assuming $R > 1$, prove that for all $r \in [1, R)$, the following identity holds:

$$\int_{\partial B_1} u(r\sigma)u(r^{-1}\sigma)\,\mathrm{d}\sigma = \int_{\partial B_1} u(\sigma)^2\,\mathrm{d}\sigma.$$

(b) Prove that for any $0 < r_1 < r_2 < R$, if we let $s = \sqrt{r_1 r_2}$, then

$$\int_{\partial B_1} u(r_1\sigma)u(r_2\sigma)\,\mathrm{d}\sigma = \int_{\partial B_1} u(s\sigma)^2\,\mathrm{d}\sigma.$$

Hint. *Apply the part (a) to $u(sx)$ and $r = r_2/s$.*

(c) Prove that for any $0 < r_1 < r_2 < R$, if we let $s = \sqrt{r_1 r_2}$, then

$$\left(\int_{\partial B_1} u(r_1\sigma)^2\,\mathrm{d}\sigma\right)\cdot\left(\int_{\partial B_1} u(r_2\sigma)^2\,\mathrm{d}\sigma\right) \ge \left(\int_{\partial B_1} u(s\sigma)^2\,\mathrm{d}\sigma\right)^2.$$

(d) Explain how the previous inequality can be used to prove the unique continuation principle for harmonic functions without using analyticity.

Exercise∗ 4.28 (Almgren's frequency formula)**.** *Under the same assumptions as in Question 4.27, let us define*

$$\begin{aligned} N(r) &:= \int_{\partial B_r} u\, u_v\,\mathrm{d}S = \int_{B_r} |\nabla u|^2\,\mathrm{d}x,\\ D(r) &:= \int_{\partial B_r} u^2\,\mathrm{d}S,\\ F(r) &= \log D(e^t). \end{aligned}$$

(a) Prove that F is convex (hint: use Hadamard's three-circle theorem).

(b) Verify the following identity

$$D' = \frac{d-1}{r}D + 2N.$$

(c) Prove that the following quantity is monotone increasing with respect to r:

$$A(r) := \frac{rN(r)}{D(r)}.$$

Exercise∗ 4.29. *Let Ω be a connected open set and u be harmonic and non-constant in Ω. Prove that u is an open map. That means that for any open subset $G \subset \Omega$, its corresponding image $u(G)$ is an open set in $\mathbb{R}$.*

Exercise∗ 4.30. *Let u be harmonic and positive outside of the ball $\overline{B}_1$. Prove that there exists a positive constant $c > 0$ so that*

$$u(x) \ge \begin{cases} c\left(1 - |x|^{2-d}\right) & \textit{for } d \ge 3,\\ c\log|x| & \textit{for } d = 2. \end{cases}$$

Hint. Use Questions 4.8 and 4.10.

Exercise∗ 4.31. *(a) Let* $u : \mathbb{R}^d \to [0, \infty)$ *be a continuous nonnegative function such that* $\Delta u = 0$ *outside* $\overline{B}_1$ *and* $u = 0$ *in* $\overline{B}_1$*. Prove that for some constant* $c \geq 0$*,*

$$u(x) = \begin{cases} c\left(1 - |x|^{2-d}\right) & \text{for } d \geq 3, \\ c \log |x| & \text{for } d = 2. \end{cases}$$

(b) Let $\Lambda \subset \mathbb{R}^d$ *be a compact set. Let* $u_1, u_2 : \mathbb{R}^d \to [0, \infty)$ *be two continuous nonnegative functions such that* $\Delta u_1 = \Delta u_2 = 0$ *outside* Λ*, and* $u_1 = u_2 = 0$ *in* Λ*. Prove that one of these functions is a scalar multiple of the other.*

Chapter 5

The Wave equation

We turn our attention to the wave equation. We start by computing explicit formulas for the solutions of the wave equation in the full space in dimensions one, two and three. Later, we explore some qualitative properties of the solutions, including the conservation of energy and the finite speed of propagation.

5.1 The wave equation in 1D: D'Alembert formula

5.1.1 The equation with a zero right hand side

We analyze the wave equation in one space dimension:

$$\begin{aligned} u_{tt} - u_{xx} &= 0, \\ u(0,x) &= f(x), \\ u_t(0,x) &= g(x). \end{aligned}$$

It is convenient to rewrite the equation in the following form

$$(\partial_t - \partial_x)(\partial_t + \partial_x)u = u_{tt} - u_{xx} = 0.$$

Let us define $v := (\partial_t + \partial_x)u$. The function v satisfies the simple transport equation $v_t - v_x = 0$, thus

$$v(t,x) = v(0,x+t) = g(x+t) + f'(x+t).$$

Likewise, we can also write

$$(\partial_t + \partial_x)(\partial_t - \partial_x)u = u_{tt} - u_{xx} = 0.$$

Let us define $w := (\partial_t - \partial_x)u$. The function w satisfies the simple transport equation $w_t + w_x = 0$. Let us recall that this transport equation has an explicit solution: $w(t,x) = w(0,x-t)$. Therefore,

$$w(t,x) = w(0,x-t) = g(x-t) - f'(x-t).$$

We now observe that

$$u_t(t,x) = \frac{v(t,x) + w(t,x)}{2} = \frac{g(x-t) + g(x+t)}{2} + \frac{f'(x+t) - f'(x-t)}{2}.$$

We obtain a formula for $u(t,x)$ integrating the values of u_t.

$$\begin{aligned} u(t,x) &= u(0,x) + \int_0^t u_t(s,x) \, \mathrm{d}s, \\ &= f(x) + \int_0^t \frac{g(x-s) + g(x+s)}{2} + \frac{f'(x+s) - f'(x-s)}{2} \, \mathrm{d}s, \end{aligned}$$

Applying the fundamental theorem of calculus to the second term inside the integral,

$$
\begin{aligned}
&= f(x) + \frac{f(x+t)+f(x-t)-2f(x)}{2} + \frac{1}{2}\int_{-t}^{t} g(x+s)\,\mathrm{d}s, \\
&= \frac{f(x+t)+f(x-t)}{2} + \frac{1}{2}\int_{-t}^{t} g(x+s)\,\mathrm{d}s.
\end{aligned}
$$

This last expression is known as D'Alembert formula. It allows us to compute the solution to the wave equation in the full real line. A byproduct of the derivation of the formula is that this solution must be unique.

5.1.2 The wave equation in 1D with a non-zero right hand side

We now want to derive a formula to solve the wave equation with a non-zero right hand side

$$
\begin{aligned}
u_{tt} - u_{xx} &= h(t,x), \\
u(0,x) &= f(x), \\
u_t(0,x) &= g(x).
\end{aligned}
$$

In order to apply Duhamel principle, we have the small inconvenience that our equation is of second order in time. We apply the usual trick to transform a second order equation into a first order system by writing an evolution equation for the pair (u, u_t). That is, we write a system of equations for a pair of functions that we write u and v. We build this system so that u is the solution to the wave equation and $v = u_t$. The system reads

$$
\begin{aligned}
u_t - v &= 0, \\
v_t - u_{xx} &= h(t,x), \\
u(0,x) &= f(x), \\
v(0,x) &= g(x).
\end{aligned}
$$

This is now a system of equations involving only one derivative in time. In terms of the abstract setting of Duhamel formula, we have $\mathcal{A}(u,v) = (v, u_{xx})$. Introducing the right hand side $(0,h)$ to the equation for $(u,v) = (u,u_t)$ with the solution operator S_t given in D'Alembert formula, we end up with

$$
u(t,x) = \frac{f(x+t)+f(x-t)}{2} + \frac{1}{2}\int_{-t}^{t} g(x+s)\,\mathrm{d}s + \frac{1}{2}\int_0^t \int_{r-t}^{t-r} h(r, x+s)\,\mathrm{d}s\,\mathrm{d}r.
$$

5.1.3 The wave equation on half lines

We analyze now how to extend D'Alembert formula to solutions of the wave equation in the half line $(0,+\infty)$ with either Dirichlet or Neumann conditions on $x = 0$.

Let us start by analyzing the Dirichlet case. We want to solve the equation

$$
\begin{aligned}
u_{tt} - u_{xx} &= 0 \;\text{ for } t,x \in (0,+\infty)\times(0,+\infty), \\
u(t,0) &= 0 \text{ for } t \in (0,+\infty), \\
u(0,x) &= f(x) \text{ for } x \in (0,+\infty), \\
u_t(0,x) &= g(x) \text{ for } x \in (0,+\infty).
\end{aligned}
$$

Let us assume that $f(0) = 0$ and $g(0) = 0$. In order to transform this problem into one in the full line, we make an odd reflection (which is compatible with the Dirichlet condition). Let $\tilde{f}$ and $\tilde{g}$ be an odd extension of f and g. That is, we define

$$
\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in [0,+\infty), \\ -f(-x) & \text{if } x \in (-\infty,0). \end{cases}
$$

$$
\tilde{g}(x) = \begin{cases} g(x) & \text{if } x \in [0,+\infty), \\ -g(-x) & \text{if } x \in (-\infty,0). \end{cases}
$$

We solve the wave equation in the full line with $\tilde{f}$ and $\tilde{g}$ as boundary conditions.

$$\begin{aligned}\tilde{u}_{tt} - \tilde{u}_{xx} = 0 \quad &\text{for } t, x \in (0, +\infty) \times \mathbb{R},\\ \tilde{u}(0,x) = \tilde{f}(x) \ &\text{for } x \in \mathbb{R},\\ \tilde{u}_t(0,x) = \tilde{g}(x) \ &\text{for } x \in \mathbb{R}.\end{aligned}$$

Note that if $\tilde{u}(t,x)$ is a solution, then also $-\tilde{u}(t,-x)$ is a solution to the same equation. Since the wave equation has a unique solution, then $\tilde{u}(t,x) = -\tilde{u}(t,-x)$. This means that $\tilde{u}$ is an odd function in x for all t. The fact that $u(t,x)$ is odd in x implies that $\tilde{u}(t,0) = 0$ for all t. Therefore the restriction of $\tilde{u}$ to $[0,+\infty) \times [0,+\infty)$ has to be the unique solution u of the wave equation in the half line with the Dirichlet condition $u(t,0) = 0$.

Applying D'Alembert formula for the equation for $\tilde{u}$, we derive the formula, for $t, x \in (0,\infty) \times (0,\infty)$,

$$\begin{aligned}u(t,x) &= \frac{\tilde{f}(x-t) + \tilde{f}(x+t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} \tilde{g}(y)\,\mathrm{d}y,\\ &= \begin{cases} \frac{f(x-t)+f(x+t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} g(y)\,\mathrm{d}y & \text{if } x > t,\\ \frac{f(x+t)-f(t-x)}{2} + \frac{1}{2}\int_{t-x}^{x+t} g(y)\,\mathrm{d}y & \text{if } x < t.\end{cases}\end{aligned}$$

For the last identity, we used that

$$\int_{x-t}^{t-x} \tilde{g}(y)\,\mathrm{d}y = 0,$$

since $\tilde{g}$ is odd.

If we want to derive a similar formula for the wave equation in the half line but with the Neumann boundary condition $u_x(t,0)$ instead of the Dirichlet boundary condition $u(t,0) = 0$, we can repeat a similar analysis using an even reflection instead of an odd reflection.

5.2 Wave equation in 3D : spherical means

We study solutions of the equation

$$u_{tt} = \Delta u,$$

where $u(t,x)$ is a function $u : [0,\infty) \times \mathbb{R}^d \to [0,+\infty)$ with given initial data

$$\begin{aligned}u(0,x) &= f(x),\\ u_t(0,x) &= g(x).\end{aligned}$$

We will consider the spherical means

$$U(t,r) := \frac{1}{|\partial B_r|}\int_{\partial B_r(x)} u(t,y)\,\mathrm{d}S(y).$$

We will derive an equation for U. For that, we compute U_{tt}, U_r and U_{rr}. We start with the observation that

$$\begin{aligned}U(t,r) &= \frac{1}{|\partial B_r|}\int_{\partial B_1} u(t, x + r\theta) r^{d-1}\,\mathrm{d}S(\theta),\\ &= \frac{1}{|\partial B_1|}\int_{\partial B_1} u(t, x + r\theta)\,\mathrm{d}S(\theta).\end{aligned}$$

Thus, we obtain

$$\begin{aligned}U_r(t,r) &= \frac{1}{|\partial B_1|}\int_{\partial B_1} u_r(t, x + r\theta)\,\mathrm{d}S(\theta),\\ &= \frac{1}{|\partial B_r|}\int_{\partial B_r(x)} u_\nu(t,y)\,\mathrm{d}S(y),\\ &= \frac{1}{r^{d-1}|\partial B_1|}\int_{B_r(x)} \Delta u(t,y)\,\mathrm{d}y, \qquad \text{applying the divergence theorem.}\end{aligned}$$

We differentiate this last right-hand side in order to obtain an expression for U_{rr}.

$$\begin{aligned} U_{rr}(t,r) &= -\frac{d-1}{r^d|\partial B_1|}\int_{B_r(x)} \Delta u(t,y)\,\mathrm{d}y + \frac{1}{r^{d-1}|\partial B_1|}\int_{\partial B_r(x)} \Delta u(t,y)\,\mathrm{d}S(y), \\ &= -\frac{d-1}{r}U_r + \frac{1}{|\partial B_r|}\int_{\partial B_r(x)} \Delta u\,\mathrm{d}S \end{aligned}$$

Moreover,

$$\begin{aligned} U_{tt} &= \frac{1}{|\partial B_r|}\int_{\partial B_r(x)} u_{tt}\,\mathrm{d}S, \\ &= \frac{1}{|\partial B_r|}\int_{\partial B_r(x)} \Delta u\,\mathrm{d}S \end{aligned}$$

Therefore

$$U_{tt} = U_{rr} + \frac{d-1}{r}U_r.$$

We could have deduced this final formula $U_{tt} = U_{rr} + \frac{d-1}{r}U_r$ by realizing that the function $U(|x-x_0|,t)$ is the average of all rotations of u around the point x_0. Thus, it is a radially symmetric function which solves the wave equation (the formula for U is the wave equation written in polar coordinates).

It turns out that in space dimension three ($d = 3$), the equation for U can be reduced to the wave equation in 1D. Indeed, we see that

$$(rU)_{tt} = rU_{tt} = rU_{rr} + 2U_r = (rU)_{rr}.$$

Therefore, the function $rU(t,r)$ solves the 1D wave equation for $r > 0$ and $t > 0$. Obviously, $rU(t,r) = 0$ if $r = 0$, we can then apply the formula from the previous section. Let F and G be the spherical averages of f and g respectively,

$$\begin{aligned} F(r) &:= \frac{1}{|\partial B_r|}\int_{\partial B_r(x)} f(y)\,\mathrm{d}S(y), \\ G(r) &:= \frac{1}{|\partial B_r|}\int_{\partial B_r(x)} g(y)\,\mathrm{d}S(y) \end{aligned}$$

Then we clearly have $rU(0,r) = rF(r)$ and $rU_t(0,r) = rG(r)$. We apply D'Alembert formula for the half line to obtain

$$rU(t,r) = \frac{(t+r)F(t+r) - (t-r)F(t-r)}{2} + \frac{1}{2}\int_{t-r}^{t+r} sG(s)\,\mathrm{d}s, \quad \text{if } r < t.$$

In order to compute $u(t,x)$ we have to take the limit as $r \to 0$ in the expression above,

$$\begin{aligned} u(t,x) &= \lim_{r\to 0}\frac{rU(r)}{r}, \\ &= \lim_{r\to 0}\frac{(t+r)F(t+r) - (t-r)F(t-r)}{2r} + \frac{1}{2r}\int_{t-r}^{t+r} sG(s)\,\mathrm{d}s, \\ &= \frac{\partial}{\partial t}\left[tF(t)\right] + tG(t), \\ &= \frac{\partial}{\partial t}\left(\frac{1}{4\pi t}\int_{\partial B_t(x)} f(y)\,\mathrm{d}S(y)\right) + \frac{1}{4\pi t}\int_{\partial B_t(x)} g(y)\,\mathrm{d}S(y) \end{aligned}$$

This is the formula which solves the wave equation in 3D with initial data f and g. Note that the value of $u(t,x)$ depends only on the values of f and g on the sphere of radius t centered at x (actually an arbitrarily small neighborhood of it for f).

5.2.1 The wave equation in 2D

We can use the formula for the wave equation in 3D developed in the previous section to derive a formula in dimension two. Let us suppose we want to solve the wave equation in 2D. That is, we look for a function $u : [0, +\infty) \times \mathbb{R}^2 \to \mathbb{R}$ such that

$$\begin{aligned} u_{tt} - \Delta u &= 0, \\ u(0, x) &= f(x), \\ u_t(0, x) &= g(x). \end{aligned}$$

In order to solve this problem, we extend f, g to $\mathbb{R}^3$ simply by making them independent of the third variable: $\tilde{f}(x_1, x_2, x_3) = f(x_1, x_2)$ and $\tilde{g}(x_1, x_2, x_3) = g(x_1, x_2)$. We solve the problem in 3D with initial data $\tilde{f}$ and $\tilde{g}$ which gives us a solution $\tilde{u}$ which is independent of the variable x_3, and there is an explicit formula for $\tilde{u}$.

$$u(t, x_1, x_2) = \tilde{u}(t, x_1, x_2, 0) = \frac{\partial}{\partial t}\left(\frac{1}{4\pi t}\int_{\partial B_t(x)} \tilde{f}(y) \,\mathrm{d}S(y)\right) + \frac{1}{4\pi t}\int_{\partial B_t(x)} \tilde{g}(y) \,\mathrm{d}S(y),$$

(The spheres ∂B_t above are inside $\mathbb{R}^3$)

$$= \frac{\partial}{\partial t}\left(\frac{1}{2\pi}\int_{B_t(x)} f(y)\frac{1}{\sqrt{t^2 - |x-y|^2}} \,\mathrm{d}y\right) + \frac{1}{2\pi}\int_{B_t(x)} g(y)\frac{1}{\sqrt{t^2 - |x-y|^2}} \,\mathrm{d}y,$$

(The balls B_t above are discs in $\mathbb{R}^2$)

5.3 Uniqueness and Finite speed of propagation

We start with a statement about the conservation of energy for a suitable class of solutions.

Theorem 5.3.1. *Let $U \subset \mathbb{R}^d$ be a bounded domain with a piecewise C^1 boundary. Let $u : [0, T] \times U \to \mathbb{R}$ be a solution to the wave equation in $(0, T) \times U$. Assume that u is C^2 and either $u = 0$ on $[0, T] \times \partial U$ (Dirichlet boundary condition) or $u_\nu = 0$ on $[0, T] \times \partial U$ (Neumann boundary condition). Then, the energy*

$$E(t) := \int_U |\nabla u|^2 + |u_t|^2 \,\mathrm{d}x,$$

is a constant function of t.

Proof. We differentiate inside the integral to compute

$$E'(t) = 2\int_U \nabla u \cdot \nabla u_t + u_t\, u_{tt} \,\mathrm{d}x,$$

We integrate by parts the first term using the divergence theorem.

$$E'(t) = 2\int_U -\Delta u\, u_t + u_t\, u_{tt} \,\mathrm{d}x + \int_{\partial U} u_\nu u_t \,\mathrm{d}A,$$

In the case of Dirichlet boundary conditions, the function u is constant zero for all values of (t, x) so that $x \in \partial U$. Thus, the factor u_t vanishes on ∂U. In the case of Neumann boundary conditions, we immediately have that $u_\nu = 0$ on ∂U.

$$\begin{aligned} &= 2\int_U u_t(u_{tt} - \Delta u) \,\mathrm{d}x, \\ &= 0. \end{aligned}$$

□

A corollary of the conservation of energy is the uniqueness of solutions of the wave equation in bounded domains. Indeed, if u_1 and u_2 are two solutions with identical initial data, then $w = u_1 - u_2$ would be another solution of the wave equation with zero initial data. The energy corresponding to w would vanish. In particular w_t would be identically zero, and therefore $w(t,x)$ would be zero for all $t \geq 0$ and $x \in \mathbb{R}$.

The next theorem tells us that the energy restricted to the cone of influence of a point (t_0, x_0) is non-increasing. From the following theorem, we will deduce the finite speed of propagation of information, and the uniqueness of solutions without any restriction of their behavior at infinity.

Theorem 5.3.2. *Let $u : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ be a solution to the wave equation in $(0,\infty) \times \mathbb{R}^d$. Assume that u is smooth and (t_0, x_0) is an arbitrary point in $(0,\infty) \times \mathbb{R}^d$. Let*

$$E(t) := \int_{B_{t_0 - t}(x_0)} |\nabla u|^2 + |u_t|^2 \, \mathrm{d}x.$$

Then $E : [0, t_0] \to \mathbb{R}$ is non increasing with respect to t.

Proof. We differentiate $E(t)$.

$$E'(t) = -\int_{\partial B_{t_0 - t}(x_0)} |\nabla u|^2 + |u_t|^2 \, \mathrm{d}S + 2 \int_{B_{t_0 - t}(x_0)} \nabla u \cdot \nabla u_t + u_t \, u_{tt} \, \mathrm{d}x,$$

We integrate by parts the second term as in the proof of Theorem 5.3.1. Some boundary terms are produced this time.

$$= -\int_{\partial B_{t_0 - t}(x_0)} |\nabla u|^2 + |u_t|^2 - 2u_t u_\nu \, \mathrm{d}S + 2 \int_{B_{t_0 - t}(x_0)} u_t(u_{tt} - \Delta u) \, \mathrm{d}x,$$

The second integral vanishes because of the equation. We use that $|\nabla u| \geq |u_\nu|$ to estimate the first term,

$$\leq -\int_{\partial B_{t_0 - t}(x_0)} |u_\nu|^2 + |u_t|^2 - 2u_t u_\nu \, \mathrm{d}S = -\int_{\partial B_{t_0 - t}(x_0)} (u_\nu - u_t)^2 \, \mathrm{d}S \leq 0.$$

Since $E'(t) \leq 0$ for all t, then E is non-increasing. □

Corollary 5.3.3. *Suppose that two pairs of initial conditions (f_1, g_1) and (f_2, g_2) coincide on a ball $B_r(x_0)$. Then, the corresponding solutions u_1 and u_2 of the wave equation with initial data (f_1, g_1) and (f_2, g_2) respectively, coincide in the cone*

$$\{(t,x) : t \in [0,r], |x - x_0| \leq r - t\}.$$

Proof. Set $(t_0, x_0) = (r, x_0)$ and apply Theorem 5.3.2 to the function $w = u_1 - u_2$. □

In particular, applying the previous corollary to the case $f = f_1 = f_2$ and $g = g_1 = g_2$ we conclude the uniqueness of solutions.

Corollary 5.3.4. *For any pair of smooth functions $f, g : \mathbb{R}^d \to \mathbb{R}$, there is at most one solution to the initial value problem for the wave equation.*

$$\begin{aligned} u_{tt} &= \Delta u && \text{in } [0,T] \times \mathbb{R}^d, \\ u(0,x) &= f(x) && \text{for } x \in \mathbb{R}^d, \\ u_t(0,x) &= g(x) && \text{for } x \in \mathbb{R}^d. \end{aligned}$$

Comparing this uniqueness result with the corresponding one for the heat equation in Corollary 3.2.4, we observe that for the wave equation, we do not need to assume that the function u is bounded. In fact, there is no restriction on the decay or growth of u, f or g at infinity. The value of u at (t,x) depends only of the values of f and g within the ball $B_t(x)$.

5.4 Exercises

Exercise 5.1. *Prove that any function $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ that solves the wave equation $u_{tt} = u_{xx}$ (in 1D) can be written as $u(t,x) = F(x-t) + G(x+t)$ for some pair of functions $F, G : \mathbb{R} \to \mathbb{R}$.*

Exercise 5.2. *Let u be a solution to the wave equation $u_{tt} - u_{xx} = 0$ in $\mathbb{R} \times [0,\infty)$. Assume that both $u(0,x)$ and $u_t(0,x)$ are smooth and compactly supported. Let*

$$E_{kin}(t) = \int u_t^2 \, \mathrm{d}x,$$
$$E_{pot}(t) = \int u_x^2 \, \mathrm{d}x.$$

Show that there exists T such that for all $t \geq T$, $E_{kin}(t) = E_{pot}(t)$.

Exercise 5.3. *Derive a formula for the solution to the wave equation in $[0,\infty) \times \mathbb{R}^3$ with non-zero right hand side.*

Exercise 5.4. *Let u be a solution to the wave equation*

$$\begin{aligned} u_{tt}(t,x) &= \Delta u(t,x), \text{ for } x \in \mathbb{R}^d, t > 0, \\ u(0,x) &= f(x), \text{ for } x \in \mathbb{R}^d, \\ u_t(0,x) &= g(x), \text{ for } x \in \mathbb{R}^d. \end{aligned}$$

Prove that the function $\tilde{u}(t,x) = u(ct,x)$ is a solution to the equation

$$\begin{aligned} \tilde{u}_{tt}(t,x) &= c^2 \Delta \tilde{u}(t,x), \text{ for } x \in \mathbb{R}^d, t > 0, \\ \tilde{u}(0,x) &= f(x), \text{ for } x \in \mathbb{R}^d, \\ \tilde{u}_t(0,x) &= cg(x), \text{ for } x \in \mathbb{R}^d. \end{aligned}$$

Note: the previous exercise shows that a mathematical understanding of the wave equation with speed one ($c = 1$) is enough to understand the general case of any speed c after a change of variables.

Exercise 5.5. *Let $u : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ be a solution to the wave equation in 1D.*

$$\begin{aligned} u_{tt} &= \partial_x^2 u, && \text{for } t > 0, x \in \mathbb{R} \\ u(0,x) &= f(x), \\ u_t(0,x) &= g(x). \end{aligned}$$

(a) Prove that for all $t > 0$ and $x \in \mathbb{R}$

$$\partial_x u(t,x) \leq \max |f'| + \max |g|.$$

(b) Prove that for all $t > 0$, $x \in \mathbb{R}$ and $k \in \mathbb{N}$

$$\partial_x^k u(t,x) \leq \max |f^{(k)}| + \max |g^{(k-1)}|.$$

Exercise 5.6. *Let u and v be the solution of the following wave equations in $[0,\infty) \times \mathbb{R}^d$.*

$$\begin{aligned} u_{tt} - \Delta u = v_{tt} - \Delta v &= 0, \\ u(0,x) = v_t(0,x) &= f(x), \\ u_t(0,x) = v(0,x) &= 0. \end{aligned}$$

Prove that $u = v_t$.

Exercise 5.7.

a *Let u be a solution to the following equation*

$$u_{tt} = u_{xx} - u.$$

Assume that $u(0,x) = u_t(0,x) = 0$ for all $x \in [a,b]$. Prove that $u(t,x) = 0$ if $a + t < x < b - t$.

b *Let u be a solution to the following equation*

$$u_{tt} = u_{xx} - u^3.$$

Assume that $u(0,x) = u_t(0,x) = 0$ for all $x \in [a,b]$. Prove that $u(t,x) = 0$ if $a + t < x < b - t$.

Exercise 5.8. *Let $u : [0,\infty) \times \mathbb{R}^3 \to \mathbb{R}$ solve*

$$\begin{aligned} u_{tt} &= \Delta u, \qquad \text{for } t > 0, x \in \mathbb{R}^3, \\ u(0,x) &= 0, \\ u_t(0,x) &= g(x). \end{aligned}$$

Suppose g is supported in a ball of radius R, and moreover $|g(x)| \leq M$ for every $x \in B_R$. Prove that

$$|u(t,x)| \leq \frac{MR^2}{t}.$$

Exercise 5.9. *Let $u : [0,\infty) \times B_1 \to \mathbb{R}$ be a solution of*

$$\begin{aligned} u_{tt} - \Delta u &= 1 \qquad \text{for } t > 0 \text{ and } x \in B_1, \\ u_\nu &= 0 \qquad \text{for } t > 0 \text{ and } x \in \partial B_1, \\ u(0,x) &= f(x), \\ u_t(0,x) &= g(x). \end{aligned}$$

Assume that

$$\int_{B_1} g(x) \, \mathrm{d}x = 0.$$

Let us consider the energy

$$E(t) := \int_{B_1} u_t^2 + |\nabla u|^2 \, \mathrm{d}x.$$

Prove that $E(t) = E(0) + t^2 |B_1|$ for any value of t.

Exercise 5.10. *Let $u : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ solve the wave equation with initial data f and g. Assume that f and g are compactly supported*

(a) Show that $u(t,\cdot)$ is compactly supported for all $t > 0$.

(b) Prove that $\int_{\mathbb{R}^d} u_t(t,x) \, \mathrm{d}x$ is constant in time.

(c) Conclude that $\int_{\mathbb{R}^d} u(t,x) \, \mathrm{d}x = \int_{\mathbb{R}^d} f(x) \, \mathrm{d}x + t \int_{\mathbb{R}^d} g(x) \, \mathrm{d}x$.

Exercise 5.11. *Let u be a solution of the wave equation with variable coefficients*

$$u_{tt} - \operatorname{div}[c(x)^2 \nabla u] = 0.$$

Assume that $0 \leq c(x) \leq s$ for every value of x. For any fixed point (t_0, x_0), we define

$$E(t) := \int_{B_{s(t_0 - t)}(x_0)} u_t^2 + c(x)^2 |\nabla u|^2 \, \mathrm{d}x.$$

Prove that $E(t)$ is non-increasing for $t \in [0, t_0]$.

Exercise 5.12.

(a) For any function $g : \mathbb{R}^3 \to \mathbb{R}$, which is C^1 and compactly supported, show that

$$\int_{\partial B_r} g \,\mathrm{d}A = -r^2 \int_{\mathbb{R}^3 \setminus B_r} \nabla g(y) \cdot \frac{y}{|y|^3} \,\mathrm{d}y.$$

Hint. *For $\sigma \in \partial B_1$, use the fundamental theorem of calculus to verify that that*

$$g(r\sigma) = -\int_r^\infty \nabla g(s\sigma) \cdot \sigma \,\mathrm{d}s.$$

Note. *A posteriori, the equality holds for any C^1 function g for which the expression in the right hand side is integrable.*

(b) Let $u : [0, \infty) \times \mathbb{R}^3 \to \mathbb{R}$ solve

$$\begin{aligned} u_{tt} &= \Delta u, \qquad \text{for } t > 0, x \in \mathbb{R}^3, \\ u(0,x) &= 0, \\ u_t(0,x) &= g(x). \end{aligned}$$

Prove that

$$u(t,x) \leq \frac{C}{t} \int_{\mathbb{R}^3} |\nabla g(y)| \,\mathrm{d}y.$$

Exercise 5.13. *Let $u : [0, \infty) \times \mathbb{R}^3 \to \mathbb{R}$ solve*

$$\begin{aligned} u_{tt} &= \Delta u, \qquad \text{for } t > 0, x \in \mathbb{R}^3, \\ u(0,x) &= g(x), \\ u_t(0,x) &= h(x). \end{aligned}$$

Prove that

$$\int_0^\infty u(t,0) \,\mathrm{d}t = \int_{\mathbb{R}^3} \frac{h(x)}{4\pi|x|} \,\mathrm{d}x.$$

Exercise 5.14.

(a) Given any unit vector e and a function $f : \mathbb{R} \to \mathbb{R}$, prove that the function $u : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ given by

$$u(t,x) = f(e \cdot x - t),$$

satisfies the wave equation $u_{tt} - \Delta u = 0$.

(b) Suppose that we are given a family of functions $f_e : \mathbb{R} \to \mathbb{R}$ parametrized by $e \in \partial B_1$. Prove that the function $u : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ given by

$$u(t,x) = \int_{\partial B_1} f_e(e \cdot x - t) \,\mathrm{d}S(e),$$

satisfies the wave equation $u_{tt} - \Delta u = 0$.

*(c**) Can any solution of the wave equation be written as in part (b)?*

Exercise 5.15 (Lorentz invariance)**.** *Let $u : \mathbb{R}^{1+d} \to \mathbb{R}$ be a solution to the wave equation $u_{tt} - \Delta u = 0$. Let $b \in \mathbb{R}^d$ be an arbitrary vector so that $|b| < 1$ and $\gamma = (1 - |b|^2)^{-1/2}$. Prove that the function*

$$v(t,x) := u\left(\gamma(t - b \cdot x), x + (\gamma - 1)\frac{x \cdot b}{|b|^2} b - \gamma t b\right)$$

also solves the wave equation. More generally, given any linear transformation $T : \mathbb{R}^{1+d} \to \mathbb{R}^{1+d}$ *so that* $TLT^t = L$, *where*

$$L = \begin{pmatrix} -1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & & & & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix},$$

then the function $v(t,x) := u(T(t,x))$ *solves the wave equation.*

Chapter 6

First order equations

Up to this point, the only first order equation that we have studied is the transport equation $u_t + b \cdot \nabla_x u = 0$ for some fixed vector b. In this chapter, we study more general first order partial differential equations. The method of characteristics allows us to reinterpret first order partial differential equations as ODEs over certain carefully chosen curves (the *characteristic curves*). It works for practically any first order equation. We obtain a general existence and uniqueness result for a short period of time. It also uncovers an unavoidable difficulty: many nonlinear equations do not have solutions (in the classical sense) for all positive time. Singularities, in the form of discontinuities, are created spontaneously as a result of the collision between two characteristic curves. In order to solve an equation past the time when singularities occur, we need to interpret what it means for a possibly discontinuous function to solve a partial differential equation. To this end, we discuss generalized notions of solutions.

6.1 The method of characteristics

We discuss the method of characteristics in increasing level of generality and, consequently, in increasing level of difficulty. We start with linear equations, and then we progress toward nonlinear ones.

6.1.1 The linear transport equation.

Let us first consider the transport equation with **constant coefficients** in one space dimension.

$$u_t + bu_x = 0.$$

We look for a solution $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ which satisfies some initial value $u(0, x) = f(x)$.

We note that if we follow the straight lines, $(t, x + bt)$ the solution remains constant. Indeed, we check it taking derivative with respect to time and applying the chain rule.

$$D_t[u(t, x + bt)] = b \cdot u_x(t, x + bt) + u_t(t, x + bt) = 0.$$

Therefore, we obtain that $u(t, x + bt) = u(0, x) = f(x)$ for all values of t. In order to obtain a formula that solves the equation, we invert this relation. In this case it reads

$$u(t, x) = f(x - bt).$$

Let us now move on to the **variable coefficient** case in arbitrary dimension. We will also put a right hand side.

$$u_t + b(t, x)\nabla u = c(t, x). \tag{6.1}$$

And we are given an initial data $u(0, x) = f(x)$.

We will now compute the values of u along the *characteristic curves* of the equation. In the previous example, these curves were the straight lines $(t, x - tb)$ and u was constant along these lines. Now, we define them by the ODE

$$X'(t) = b(t, X(t)), \tag{6.2}$$
$$U'(t) = c(t, X(t)). \tag{6.3}$$

We verify that $u(t, X(t)) - U(t)$ is constant in time. Indeed, differentiating in t we get using chain rule,

$$\begin{aligned} D_t[u(t, X(t)) - U(t)] &= X' \cdot \nabla u + u_t - U', \\ &= b \cdot \nabla u + u_t - c = 0. \end{aligned}$$

Therefore, if we want to compute the solution to the equation (6.1), we only need to solve the ODE (6.2) backwards by setting $X(t_0) = x_0$ so that $u(t_0, x_0) = U(t_0)$, where $U(0) = u(0, X(0))$.

6.1.2 Quasi-linear equations

We will now study a non-linear version of the transport equation

$$u_t + b(t, x, u)\nabla u = c(t, x, u). \tag{6.4}$$

The equation is called quasi-linear because it is linear in terms of the derivatives of u but not with respect to u. This time, the value of the vector field $b(t, x, u)$ depends on u. The ODE for the characteristic curve $X(t)$ depends on the values of u on this same curve. We must write a system of ODEs that track the value of $U(t) = u(t, X(t))$ simultaneously with the position of $X(t)$.

The characteristic curve consists of a pair $X(t)$ and $U(t)$ so that

$$\begin{aligned} X(0) &= x, \\ U(0) &= u(0, x), \\ X'(t) &= b(t, X(t), U(t)), \\ U'(t) &= c(t, X(t), U(t)). \end{aligned} \tag{6.5}$$

The point x is the starting position of the characteristic curve. This ODE is engineered to have the identity $u(t, X(t)) = U(t)$. We verify it in the following proposition.

Proposition 6.1.1. *Let u be a solution of the equation* (6.4) *in* $[0, T] \times \mathbb{R}^d$. *Let* $X : [0, T] \to \mathbb{R}^d$ *and* $U : [0, T] \to \mathbb{R}$ *solve the ODE system in* (6.5). *Then* $u(t, X(t)) = U(t)$ *for all values of* $t \in [0, T]$.

Proof. Let us define the following auxiliary function $\tilde{X} : [0, T] \to \mathbb{R}^d$ and $\tilde{U} : [0, T] \to \mathbb{R}$.

$$\begin{aligned} \tilde{X}(0) &= x, \\ \tilde{X}'(t) &= b(t, \tilde{X}(t), u(t, \tilde{X}(t))), \\ \tilde{U}(t) &= u(t, \tilde{X}(t)). \end{aligned}$$

Let us compute $\tilde{U}'(t)$ using (6.4). We get

$$\begin{aligned} \tilde{U}'(t) &= \frac{\mathrm{d}}{\mathrm{d}t} u(t, \tilde{X}(t)), \\ &= u_t + \tilde{X}'(t) \cdot \nabla u, \\ &= u_t(t, \tilde{X}(t)) + b(t, \tilde{X}(t), u(t, \tilde{X}(t))) \cdot \nabla u(t, \tilde{X}(t)), \\ &= c(t, \tilde{X}(t), u(t, \tilde{X}(t))) = c(t, \tilde{X}(t), \tilde{U}(t)). \end{aligned}$$

We conclude that $\tilde{X}$ and $\tilde{U}$ satisfy exactly the same ODE as X and U. Thus, they must be the same functions because of the uniqueness of solutions of ODEs. We deduce that $U(t) = \tilde{U}(t) = u(t, \tilde{X}(t)) = u(t, X(t))$. □

The new difficulty, compared to the linear transport equation, is that it is now difficult to solve the ODE backwards. The characteristic curve depends on the value of u. For any given t_0 and x_0 we would set $X(t_0) = x_0$, but we also need to prescribe a value to $U(t_0)$ in order to solve the ODE. The value of $U(t_0) = u(t_0, x_0)$ is precisely what we are trying to compute.

We only prove that the equation (6.4) has a solution in a small neighborhood of a point.

Theorem 6.1.2. *Let x_0 be any point in $\mathbb{R}^d$. There exists an $r > 0$ (which may be small) so that there exists a unique function $u : B_r(x_0) \times (-r, r) \to \mathbb{R}$ which solves the equation*

$$\begin{aligned} u_t + b(t,x,u)\nabla u &= c(t,x,u) \quad \text{in } B_r(x_0) \times (-r,r), \\ u(0,x) &= f(x) \quad \text{for } x \in B_r(x_0). \end{aligned}$$

Proof. Since we cannot solve the ODE for the characteristic curves backward, we study the map that solves the ODE forward and prove it is invertible. We will use the inverse function theorem for that.

For every $y \in \mathbb{R}^d$ and $t \in \mathbb{R}$ we define the function $m(t,y) := (t, X(t))$ where X is the solution of the ODE for the characteristic curves with initial data y.

$$\begin{aligned} X(0) &= y, \\ X' &= b(t,x,U), \\ U(0) &= f(y), \\ U' &= c(t,x,U). \end{aligned} \tag{6.6}$$

From Picard's theorem, this map m is well defined and smooth (provided b and c are smooth).

Note that for any $x_0 \in \mathbb{R}^d$, we have $m(0,x_0) = x_0$ and $\partial_t m(0,x_0) = (1, X'(0))$. Therefore

$$Dm(0,x_0) = \begin{pmatrix} 1 & 0 \\ X'(0) & \mathrm{I} \end{pmatrix}.$$

Let us clarify our notation. By I we mean the identity matrix in $\mathbb{R}^{d\times d}$ which is an $d \times d$ sub-block of the $(d+1)\times(d+1)$ matrix Dm. The vector $X'(0)$ is an $d \times 1$ sub-block of the matrix and its values will be irrelevant for the proof. Finally, where we write '0', it is actually a $1 \times d$ arrow of zeroes. The matrix is lower triangular, and it is easy to see that $\det Dm = 1$. In particular $Dm(0,x_0)$ is invertible.

Applying the inverse function theorem, for any $x_0 \in \mathbb{R}^d$, there exists a smooth inverse m^{-1} in a neighborhood $(-r,r)\times B_r(x_0)$ of $(0,x_0)$. This means that for all $(t,x) \in (-r,r)\times B_r(x_0)$, we obtain $m^{-1}(t,x) = (t,y)$ so that when we follow the ODEs (6.6) we get $X(t) = x$. We thus can determine $u(t,x)$ as $U(t)$ for the corresponding solution U of (6.6).

We have the identity $U(t) = u(t, X(t))$ by construction. We should verify that the function u we constructed indeed satisfies the equation.

$$\begin{aligned} c(t,x,u) &= c(t, X(t), U(t)), \\ &= U'(t), \\ &= D_t[u(t,X(t))], \\ &= u_t + X' \cdot \nabla u = u_t + b \cdot \nabla u. \end{aligned}$$

Thus, indeed the function u we constructed solves the equation. □

A general existence + uniqueness theorem

Theorem 6.1.2 is a particular case of a more general result for quasilinear first-order equation. Notice that if we do not make a distinction between the space variables (x) and the time variables (t), then u_t is just like any other partial derivative in a first order equation. Thus, we can think of the equation (6.4) as a first order equation in $\mathbb{R}^{d+1}$. The initial data $u = f$ is given on any arbitrary surface S, which was chosen to be the hyperplane $t = 0$ in the previous examples. Under this interpretation, Theorem 6.1.2 is a particular case of the following more general result

Theorem 6.1.3. *Let S be a smooth $d-1$ dimensional surface in $\mathbb{R}^d$ and $x_0 \in S$. Assume $b(x_0, f(x_0))$ is not tangent to the surface S. There exists an $r > 0$ (which may be small) so that there exists a unique function $u : B_r(x_0) \to \mathbb{R}$ that solves the equation*

$$\begin{aligned} b(x,u) \cdot \nabla u &= c(x,u) \quad \text{in } B_r(x_0), \\ u(x) &= f(x) \quad \text{for } x \in S. \end{aligned}$$

The assumption that $b(x_0, f(x_0))$ is not tangent to the surface S prevents the undesirable case in which the characteristic curve $X(t)$ would stay inside the surface S. It is easy to see that in that case we would have either no solution or infinitely many.

We present the proof in the two dimensional case only in order to have a simpler notation.

Proof for the 2D case. The proof follows the same ideas as in Theorem 6.1.2. We will do only the two dimensional case. In this case, instead of a surface S we have a curve parametrized as $\gamma(s)$. Let $x_0 = \gamma(s_0)$.

For each value of s, we look at the ODE for the characteristic curves

$$\begin{aligned} X(0) &= \gamma(s), \\ X'(t) &= b(X(t), U(t)), \\ U(0) &= f(\gamma(s)), \\ U'(t) &= c(X(t), U(t)). \end{aligned}$$

Note that in this case the variable t is just a parametrization variable of the characteristic curve. There is no time variable in the equation.

We define $m(s,t)$ as the value of $X(s) \in \mathbb{R}^2$ for the solution of the ODE corresponding to this value of s. We want to prove that m has a smooth inverse function, using the inverse function theorem. We need to check that $Dm(s_0, 0)$ is invertible, in order to have a local inverse of m in a neighborhood of $(s_0, 0)$.

We have that

$$Dm(s_0, t_0) = (\gamma'(s_0) \mid b(x_0, f(x_0)))\,.$$

By this we mean that $Dm(s_0, t_0)$ is a 2×2 matrix whose columns are $\gamma'(s_0)$ and $b(x_0, f(x_0))$. The matrix is invertible provided that $b(x_0, f(x_0))$ is not parallel to $\gamma'(s_0)$, which is exactly the assumption that $b(x_0, f(x_0))$ is not parallel to the curve γ (or the surface S in the n-dimensional version of the Theorem).

Therefore, by the inverse function theorem, the map m has a differentiable inverse m^{-1} in $B_r(x_0)$, so that $m^{-1}(x) = (s,t)$. We finally define $u(x)$ as the corresponding $U(t)$ for the solution of the ODE starting at s, where s and t are given by $m^{-1}(x)$.

The verification that this function is indeed a solution is similar as in the proof of Theorem 6.1.2. □

The Burgers equation

The Burgers equation is probably the simplest example of a first order equation for which the solution obtained by the method of characteristics does not exist for all time. The equation is

$$u_t + u\, u_x = 0$$

where $u : \mathbb{R} \times (-r, r) \to \mathbb{R}$ and an initial data $u(0,x) = f(x)$ is prescribed. According to the method of characteristics, for every $x \in \mathbb{R}$, we solve the ODEs

$$\begin{aligned} X(0) &= x, \\ X' &= U, \\ U(0) &= f(x), \\ U' &= 0. \end{aligned}$$

Thus, the value of $U(t) = u(t, X(t))$ is constant along the characteristic curve. Consequently, X' is constant and $X(t)$ is a straight line.

We can see that if $f(x_1) < f(x_2)$ for some points so that $x_1 > x_2$, then the characteristic curves will cross at some positive time. Indeed, we have two characteristic curves $X_1(t) = x_1 + f(x_1)t$ and $X_2(t) = x_2 + f(x_2)t$. Certainly, the curves cross for $t = \frac{x_1 - x_2}{f(x_2) - f(x_1)}$. This means that the Burgers equation does not have any solution (at least in a classical sense) for a sufficiently large time interval.

6.1.3 Fully nonlinear first order equations

In order to illustrate the application of the method of characteristics to fully non linear first order equations, let us start with the simplest example of such equations.

The equation consists in finding the function $u(t,x)$ solving

$$u_t + u_x^2 = 0,$$

which is complemented with the initial condition $u(0,x) = f(x)$.

We want to find a characteristic curve $X(t)$ where we can track the value of $U(t) = u(t, X(t))$ together with $P(t) = u_x(t, X(t))$. It turns out that if we knew the value of u_x, then we would be able the value of u for any curve X. Thus, we will choose the characteristic curves $X(t)$ aiming at being able to track the value of u_x on them.

We differentiate the equation to obtain

$$u_{xt} + 2u_x u_{xx} = 0.$$

If we write $p(t,x) = u_x(t,x)$, the equation becomes

$$p_t + 2pp_x = 0.$$

Therefore, a good choice for $X(t)$ and $P(t)$ is given by the solution to the ODE

$$\begin{aligned} X'(t) &= 2P(t), \\ P'(t) &= 0. \end{aligned}$$

We add u_x^2 to both sides of the original equation and rewrite it as

$$u_t + 2pu_x = p^2,$$

from which we deduce the ODE for U,

$$U'(t) = P(t)^2.$$

These ODEs must be accompanied by the initial conditions

$$\begin{aligned} X(0) &= x, \\ U(0) &= f(x), \\ P(0) &= f_x(x). \end{aligned}$$

These characteristic curves allow us to solve the equation for a small period of time. Just like for the Burgers equation, for a large enough time interval, the characteristic curves will cross and there will be no classical solution any longer.

In this example, the derivative u_x happens to solve a variant of Burgers equation. As we will see in the next section, differentiating any fully nonlinear equation gives us a quasi-linear equation for the derivative of the solution.

More general first order equations

Now we turn into the more general case.

$$u_t + H(t, x, u, \nabla u) = 0, \tag{6.7}$$

with a given initial value $u(0,x) = u_0(x)$.

We have already studied the method of characteristics for quasi-linear equations. In order to solve the fully nonlinear equation (6.7), we will transform the problem into a quasilinear equation of the form (6.4).

A key observation is that in our previous analysis of the equation (6.4) it was not necessary that the function u was scalar valued. A similar analysis works if u is vector valued provided that the transport directions b are the same for every component of u. We write an equation for the pair $(p, u) \in \mathbb{R}^{d+1}$, where $p = \nabla u$, of the form

$$\partial_t \begin{pmatrix} p \\ u \end{pmatrix} + b(t, x, u, p) \cdot \nabla \begin{pmatrix} p \\ u \end{pmatrix} = c(t, x, u, p). \tag{6.8}$$

Note that for every value of (t, x, u, p) we need $b(t, x, u, p) \in \mathbb{R}^d$ and $c(t, x, u, p) \in \mathbb{R}^{d+1}$.

We start by differentiating the equation (6.7). Using the chain rule, and writing $p = \nabla u$, we obtain

$$p_t + \frac{\partial H}{\partial x} + \frac{\partial H}{\partial u} p + \frac{\partial H}{\partial p} \cdot \nabla p = 0. \tag{6.9}$$

It is convenient to clarify the notation at this point. The function p depends on x and t (i.e. $p = p(t, x)$), whereas H is always evaluated at (t, x, u, p). When we write $\partial H/\partial x$, we mean the vector valued function $(\partial H/\partial x_1, \dots, \partial H/\partial x_n)$. This would be the gradient of H respect to x, but note that H depends on t, u and p too. The quantity $\partial H/\partial u$ is scalar valued, and it is multiplied times the vector $p \in \mathbb{R}^d$. When we write $\frac{\partial H}{\partial p} \cdot \nabla p$ we mean

$$\frac{\partial H}{\partial p} \cdot \nabla p = \sum_j \frac{\partial H}{\partial p_j} \cdot \frac{\partial p}{\partial x_j}.$$

Note that this is vector valued because p is.

The first part of the equation (6.8), which corresponds to the evolution of p, is given by (6.9). We have already determined that we pick $b = \frac{\partial H}{\partial p}$. We are still missing a similar equation for the evolution of u. Indeed, we see that

$$u_t + \frac{\partial H}{\partial p} \cdot \nabla u = -H(t, x, u, p) + \frac{\partial H}{\partial p} \cdot p. \tag{6.10}$$

The right hand side is simply the combination of (6.7) with the identity $p = \nabla u$.

The quasi-linear equation of the form (6.8) that we were looking for is the combination of (6.9) and (6.10). Note that this system has the form (6.8) with

$$\begin{aligned} b &= \frac{\partial H}{\partial p}, \\ c &= \begin{pmatrix} -\frac{\partial H}{\partial x} - \frac{\partial H}{\partial u} p \\ -H + \frac{\partial H}{\partial p} \cdot p \end{pmatrix} \end{aligned}$$

Using the method of characteristics, we define the curves

$$\begin{aligned} X' &= \frac{\partial H}{\partial p}(t, X(t), U(t), P(t)), & X(0) &= x, \\ P' &= -\frac{\partial H}{\partial x}(t, X(t), U(t), P(t)) - \frac{\partial H}{\partial u}(t, X(t), U(t), P(t)) P(t), & P(0) &= \nabla u_0(x), \\ U' &= -H(t, X(t), U(t), P(t)) + \frac{\partial H}{\partial p}(t, X(t), U(t), P(t)) \cdot P(t), & U(0) &= u_0(x). \end{aligned}$$

Using the theory developed previously for first order quasilinear equations, we conclude that the solution u is determined along the characterisitic curve by the relations

$$u(t, X(t)) = U(t) \quad \text{and} \quad \nabla u(t, X(t)) = P(t).$$

Applying Theorem 6.1.2, we also deduce that for any $x_0 \in \mathbb{R}^n$, the equation (6.7) has a unique solution in a neighborhood $(-r, r) \times B_r(x_0)$.

6.2 Conservation laws.

Let us consider the following problem inspired by traffic flow. We have a very long highway. A position on the highway is parametrized by the variable $x \in \mathbb{R}$. Let us think of a one way road where all cars are traveling to the right. The function $u(t,x)$ represents the density of vehicles at any time and position. With this in mind, the amount of cars at time t between two points a and b should be approximately given by the integral

$$\int_a^b u(t,x)\,\mathrm{d}x.$$

We make the modeling assumption that the average velocity of the cars depends on the density. We write it as a function $f(u)$. Thus, if we differentiate the integral above, we observe that the rate of change of the amount of cars in the interval $[a,b]$ must be given by the rate that cars are entering through the point a minus the rate that cars are exiting at the point b. They are obtained as the product of the velocity and density at these two points. We get the identity

$$\partial_t \int_a^b u(t,x)\,\mathrm{d}x = f(u(t,a))\,u(t,a) - f(u(t,b))\,u(t,b).$$

For the sake of simpler notation, let us write $F(u) := f(u)\,u$. We have

$$\partial_t \int_a^b u(t,x)\,\mathrm{d}x = F(u(t,a)) - F(u(t,b)) = -\int_a^b \partial_x F(u(t,x))\,\mathrm{d}x.$$

Since the identity holds for every interval $[a,b]$, we derive the following partial differential equation

$$u_t + \partial_x F(u) = 0.$$

It also makes sense to study a higher dimensional version of the equation above. A common situation in physics is that the rate of change of the integral of certain quantity on a fixed domain Ω is given by the flux of a function on its boundary. We can summarize it into the identity

$$\partial_t \int_\Omega u(t,x)\,\mathrm{d}x + \int_{\partial\Omega} F(u)\cdot\nu\,\mathrm{d}S = 0. \tag{6.11}$$

A function $u:[0,+\infty)\times\mathbb{R}^n\to\mathbb{R}^m$ for which (6.11) holds for any choice of Ω would be a solution of the first order PDE.

$$u_t + \operatorname{div} F[u] = 0. \tag{6.12}$$

Indeed, we deduce (6.12) from (6.11) applying the divergence theorem. Here $F:\mathbb{R}^m\to\mathbb{R}^{n\times m}$ is a given function. The equation (6.12) written with subindices looks like

$$\partial_t u_i + \sum_j \partial_j [F_{ij}(u)] = 0.$$

Even though the most important partial differential equations in mathematical physics are systems with $m>1$, we will only discuss the scalar case $m=1$. The reason goes beyond the mere simplicity of presentation. Systems of conservation laws with $m>1$ are currently poorly understood mathematically. We are able to prove some nice theorems in the scalar case, but we would not know how obtain similar results for systems. We also focus most of our analysis to $n=1$, however, it is not difficult to generalize all the results here to $x\in\mathbb{R}^n$ for arbitrary values of n.

In the simplest case $n=m=1$, the equation is

$$u_t + \partial_x[F(u)] = 0. \tag{6.13}$$

In particular, for $F(u)=u^2/2$, we recover Burgers equation

$$u_t + u\,u_x = 0.$$

For general functions F, we can also use the chain rule in (6.13) to rewrite it as

$$u_t + F'(u)\,u_x = 0.$$

We have discussed the method of characteristics to solve equations of this form. We know that for any smooth initial condition $u(0,x) = u_0(x)$ there is locally a unique solution to the equation. If we extend the characteristic curves for large values of t, we observe that these curves may cross. The method of characterstics may give us inconsistent values for $u(t,x)$ if t is large. This means that there exists no smooth solution $u(t,x)$ in the classical sense. Solutions of conservation laws naturally develop discontinuities due to the shocks between different characteristic curves. Our objective in the next sections is to describe the meaning of the notion of *solution* to the equation (6.13), for a non-differentiable function u. We start with an informal discussion when the initial data u_0 takes only two values. In a later section, we provide a more rigorous definition of weak solution.

6.2.1 The Riemann problem

Single shocks

If we allow the solutions to the equation (6.13) to have discontinuities, we must understand how these discontinuities evolve with time. Let us start by analyzing the simplest problem in which the function u is constant on each side of a curve $x = c(t)$. Let us consider the function

$$u(t,x) = \begin{cases} u_L & \text{if } x < c(t), \\ u_R & \text{if } x > c(t). \end{cases}$$

We want to determine a condition for the curve $c(t)$ so that $u(t,x)$ solves the equation (6.13) for initial data

$$u_0(x) = \begin{cases} u_L & \text{if } x < c_0, \\ u_R & \text{if } x > c_0. \end{cases} \tag{6.14}$$

The initial motivation for the study of conservation laws comes from the identity (6.11). Thus, it is natural to verify that our function u should satisfy this identity for any $\Omega = [a,b] \subset \mathbb{R}$. We must have

$$\partial_t \int_a^b u(t,x)\, \mathrm{d}x = F(u(t,a)) - F(u(t,b)). \tag{6.15}$$

In the case of the function u described above, we get

$$\partial_t \int_a^b u(t,x)\, \mathrm{d}x = (u_L - u_R)c'(t),$$
$$F(u(t,a)) - F(u(t,b)) = F(u_L) - F(u_R).$$

Therefore, we deduce the *Rankine-Hugoniot* condition

$$c'(t) = \frac{F(u_L) - F(u_R)}{u_L - u_R}.$$

This computation suggests that the following function may be a generalized solution of (6.13). For any values of u_L and u_R in $\mathbb{R}$,

$$u(t,x) = \begin{cases} u_L & \text{if } x < vt, \\ u_R & \text{if } x > vt. \end{cases} \tag{6.16}$$

where

$$v = \frac{F(u_L) - F(u_R)}{u_L - u_R}.$$

Rarefaction waves

We will see that in some cases, we can construct other solutions for the same initial data as in the previous section. Let us consider the Burgers equation

$$u_t + u\, u_x = 0,$$

with initial data

$$u(0,x) = u_0(x) = \begin{cases} 1 \text{ if } x > 0, \\ 0 \text{ if } x < 0. \end{cases}$$

We observe that the following is a solution

$$u(t,x) = \begin{cases} 1 & \text{if } x > t, \\ x/t & \text{if } 0 \leq x \leq t, \\ 0 & \text{if } x < 0. \end{cases} \tag{6.17}$$

Indeed, this function u is continuous. It satisfies the equation classically in each of the three subdomains $\{x > t\}$, $\{0 < x < t\}$ and $\{x < 0\}$. The derivatives u_t and u_x have a jump discontinuity on the boundary of these domains. It is easy to verify that the identity (6.15) holds in this case.

For the same initial value u_0, we also have a pure shock solution as in (6.16). Thus, we must come up with a criteria to choose one of the two possible solutions. The Lax entropy condition that we will see in the next section, justify our choice of the rarefaction wave solution (6.17) over the shock as in (6.16).

Entropy condition

It is possible to construct a rarefaction wave solution similar to the one above for any initial data of the form (6.14) with $u_R > u_L$. It turns out that if $u_R < u_L$, the shock solution is the only solution to the problem. The entropy condition that we describe below will rule out the shock solution with $u_R > u_L$ in favor of the rarefaction wave solution.

If we analize the characteristic curves of the solution u in all cases above, we see a qualitative difference of the shocks when $u_R > u_L$ or $u_L > u_R$. When $u_L > u_R$, the characteristic curves flow into the shock curve $x = vt$. When $u_R > u_L$, the characteristic curves of the equation are emanated out of the shock $x = vt$. Lax's entropy condition consists in imposing the restriction that no characteristic curve should be generated on the shock for $t > 0$. This follows the intuitive idea that all information about the solution should come from the initial values. Lax's entropy solution is imposing an order of causality. Concretely, if the solution u of (6.13) is discontinuous on a curve $x = c(t)$, with left limit u_L and right limit u_R, we require that

$$F'(u_L) \geq c'(t) \geq F'(u_R). \tag{6.18}$$

Note that $F'(u_L)$ is the speed of the characteristic curve coming from the left, $F'(u_R)$ is the speed of the characteristic curve coming from the right, and $c'(t)$ is the speed of the shock. This condition tells us that characteristics are allowed to flow into the shock and never out of it. The restriction (6.18) is due to Peter Lax and known as *Lax entropy condition.*

The entropy condition serves as a criteria to select the right solution to the equation. In this case, the general philosophy is that whenever it is possible to open up a shock discontinuity as a rarefaction wave, then it should happen. Lax entropy condition (6.18) is enough to single out a unique solution to the Riemann problem for the Burgers equation, and also for several other conservation laws. [1] For general equations, it turns out that we need a more restrictive entropy condition. What we postulate is that whenever it is possible to break a shock discontinuity into a combination of smaller shocks and rarefaction waves, it should also happen. The only stable shocks are those which cannot be broken. Let us express this criteria more precisely. We consider the single shock solution (6.16) of the Riemann problem discussed above. We want to check if there is any way to find another weak solution corresponding to two smaller shocks emanating from $t = 0$, $x = 0$. If such an alternative weak solution exists, then the single shock solution is considered unstable and it should be ruled out. We want to know if there exists a value v in between u_L and u_R such that there is a two-shock solution of the form

$$u(t,x) = \begin{cases} u_L & \text{if } x < c_1 t, \\ v & \text{if } c_1 t < x < c_2 t, \\ u_R & \text{if } c_2 t < x. \end{cases} \tag{6.19}$$

[1] Precisely, (6.18) is a complete characterization of entropy solutions in the case that F is a convex function.

Both shocks must satisfy the Rankine-Hugoniot condition, thus

$$c_1 = \frac{F(v) - F(u_L)}{v - u_L} \qquad c_2 = \frac{F(u_R) - F(v)}{u_R - v}.$$

Naturally, such a solution exists provided that $c_2 > c_1$, otherwise the function u in (6.19) would be triple valued at some points.

A single shock solution as in (6.16) is considered stable when no alternative solution like (6.19) exists. Thus, we must have $c_2 \leq c_1$ for any value v between u_L and u_R, that is

$$\frac{F(v) - F(u_L)}{v - u_L} \geq \frac{F(u_R) - F(v)}{u_R - v}. \tag{6.20}$$

The value of c in (6.16) will always be in between the values of c_1 and c_2 above. Consequently, we also have

$$\frac{F(v) - F(u_L)}{v - u_L} \geq \frac{F(u_R) - F(u_L)}{u_R - u_L} \geq \frac{F(u_R) - F(v)}{u_R - v}. \tag{6.21}$$

In fact, it is not hard to verify that any of the three inequalities in (6.20) and (6.21) implies the other two.

The inequalities in (6.20) and (6.21) must hold for any intermediate value v between u_L and u_R. Note that Lax entropy condition can be recovered by taking either $v \to u_L$ or $v \to u_R$ in (6.21), so our new entropy condition is stronger.

In our analysis above, we ruled out a single shock solution whenever it can be decomposed as a two-shock solution. It does not mean that the two-shock solution will be the correct solution satisfying the entropy conditions. In fact, it can be shown that the Riemann problem with initial data corresponding to an unstable shock will always have a solution which contains a rarefaction wave, possibly combined with smaller shocks.

6.2.2 Weak and entropy solutions

In the previous section, we verified that non differentiable functions u were solutions of a conservation law by testing the identity (6.15). It is not a convenient definition. If we consider an arbitrary bounded function u, it is problematic to evaluate it at the points a and b as in (6.15) or on $\partial\Omega$ as in (6.11), since u may be discontinuous at those points. The following definition makes sense under minimal smoothness assumptions on the function u.

Definition 6.2.1. *We say that a bounded function $u : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is a weak solution of* (6.12) *if for any C^1 and compactly supported $\varphi : (0, \infty) \times \mathbb{R} \to \mathbb{R}$ we have the identity*

$$\iint u \cdot \varphi_t + F(u)\varphi_x \,\mathrm{d}x \,\mathrm{d}t = 0. \tag{6.22}$$

The purpose of the definition of weak solutions is to make sense of a partial differential equation for functions u that are not smooth. This definition is consistent with the classical notion when u is smooth. Indeed, if u satisfies (6.12) in the classical sense, then for any compactly supported function φ, we would have

$$0 = \iint (u_t + \operatorname{div} F(u))\varphi \,\mathrm{d}x \,\mathrm{d}t = -\iint u \cdot \varphi_t + F(u)\varphi_x \,\mathrm{d}x \,\mathrm{d}t. \tag{6.23}$$

The advantage of working with the second integral instead of the first is that it transfers the burden of differentiability from the solution u to the test function φ.

In some cases, the definition above can be replaced by the following simpler one. For all functions $\varphi : \mathbb{R} \to \mathbb{R}$, C^1 and compactly supported, and any two times $t_1 > t_0 \geq 0$, the following identity holds

$$\int_{\mathbb{R}} (u(t_1, x)) - u(t_0, x))\varphi(x) \,\mathrm{d}x = \int_{t_0}^{t_1} \int_{\mathbb{R}} F(u)\varphi' \,\mathrm{d}x \,\mathrm{d}t. \tag{6.24}$$

The formula (6.22) imples (6.24) provided that u satisfies a mild continuity requirement. [2] Punctilious readers may also realize that some assumption must be made already for the integral in (6.22) to be well

[2]Precisely $u(t, \cdot)$ must be weak-$*$ compact in L^1_{loc}, in particular, we can use (6.22) when $u \in C([0, \infty), L^1(\mathbb{R}))$

defined. It is most convenient to use Lebesgue integration, in which case we only need to assume that the function u is bounded and measurable (precisely $u \in L^\infty$). If we work with Riemann integrals, we would need to assume that the function u is bounded and Riemann integrable. For those readers unfamiliar with Lebesgue integration, the best attitude at this point would be to ignore integrability issues and take for granted that all functions in these notes are integrable.

Example 6.2.2. *Let $s : [0,\infty) \to \mathbb{R}$ be C^1. Assume that the function $u : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is C^1 and satisfies the equation* (6.12) *on both sides of the shock curve: $\{(t,x) : x < s(t)\}$ and $\{(t,x) : x > s(t)\}$. We claim that when the curve s satisfies the Rankine-Hugoniot condition*

$$s'(t) = \frac{F(u_R(t)) - F(u_L(t))}{u_R(t) - u_L(t)}, \qquad \text{where } u_R(t) = \lim_{x \to s(t)^+} u(t,x) \text{ and } u_L(t) = \lim_{x \to s(t)^-} u(t,x),$$

the function u will be a weak solution of (6.12)*. Let us verify the formula* (6.22) *for any admissible test function φ. We integrate by parts using the divergence theorem on boths sides of the shock curve.*

$$\iint_{x<s(t)} u\varphi_t + F(u)\varphi_x \,\mathrm{d}x\,\mathrm{d}t = -\iint_{x<s(t)} (u_t + \partial_x F(u))\varphi \,\mathrm{d}x\,\mathrm{d}t + \int_{x=s(t)} u_L\varphi n_1 + F(u_L)\varphi n_2 \,\mathrm{d}\ell,$$
$$\iint_{x>s(t)} u\varphi_t + F(u)\varphi_x \,\mathrm{d}x\,\mathrm{d}t = -\iint_{x>s(t)} (u_t + \partial_x F(u))\varphi \,\mathrm{d}x\,\mathrm{d}t - \int_{x=s(t)} u_R\varphi n_1 + F(u_R)\varphi n_2 \,\mathrm{d}\ell.$$

Here, (n_1, n_2) are the coordinates of the unit normal vector to the shock curve $x = s(t)$. We choose its orientation pointing upward. We write $\mathrm{d}\ell$ to denote differential of length. Adding the two identities above, and using the equation on both sides of the shock curve, we get

$$\iint u\varphi_t + F(u)\varphi_x \,\mathrm{d}x\,\mathrm{d}t = \int_{x=s(t)} \{(u_L - u_R)n_1 + (F(u_L) - F(u_R))n_2\}\,\varphi \,\mathrm{d}\ell.$$

We compute $n = (n_1, n_2)$ in terms of the derivative of $s(t)$. Indeed, the vector $n = \lambda(s'(t), -1)$ for some normalization scalar λ. Thus, the Rankine-Hugoniot condition means exactly that the line integral in the right hand side vanishes and we obtain

$$\iint u\varphi_t + F(u)\varphi_x \,\mathrm{d}x\,\mathrm{d}t = 0.$$

This mean, of course, that u is a weak solution of the equation (6.12)*.*

The following definition shows how we can express an inequality in the weak sense.

Definition 6.2.3. *Let u be a bounded function. We say that $u_t + \partial_x F(u) \leq 0$ weakly in some domain $\Omega \subset \mathbb{R} \times \mathbb{R}$, if for any function $\varphi \geq 0$, C^1 and compactly supported in Ω, we have*

$$\iint_\Omega (u\varphi_t + F(u)\varphi_x) \,\mathrm{d}x\,\mathrm{d}t \geq 0.$$

Note that the inequalities in the equation and in the integral expression of Definition 6.2.3 are reversed. This comes from the negative sign that one gets when integrating by parts the derivatives as in (6.23).

The purpose of Definition (6.2.1) is to encode the fact that a function u solves the equation (6.12) in a way that is verifiable even when u is not differentiable. Unfortunately, this definition is too lax in the sense that it does not distinguish between stable and unstable shocks. Like we saw in the previous sections, we need to strengthen our notion of weak solution with further entropy conditions in order to recover the uniqueness of the initial value problem. [3]

The entropy inequalities given in (6.21) make sense when the function u has a well defined jump discontinuity with values u_L and u_R on both sides of a smooth curve. If our weak solution u is merely a bounded function, it may have a rather messy discontinuous structure. A priori it may not be possible to identify

[3] It is somewhat amusing that in order to resolve the inconvenience that the notion of weak solutions is too **lax**, one resorts to entropy inequalities, the first one of which is due to Peter **Lax**.

smooth curves where the jump discontinuities take place. In order to circumvent this difficulty, we manufacture a more robust restatement of the inequalities (6.21) in a way that makes sense (and is verifiable) for arbitrary bounded functions u.

In order to motivate our next definition, let us analyze the weak solution with a single shock in Example (6.2.2). For any $v \in \mathbb{R}$, let us analyze the function $\tilde{u}(t,x) = \max(u(t,x), v)$. Note that wherever $u(t,x) > v$, we have $\tilde{u}(t,x) = u(t,x)$ and $\tilde{u}$ should satisfy the same equation as u. Also, in a set where $u(t,x) \leq v$, we will have $\tilde{u}(t,x) \equiv v$ and it also satisfies the equation (6.12). An interface between these two regions where u is continuous will not be problematic. The function $\tilde{u}$ will not satisfy the equation even weakly around those points where u has a jump discontinuity and v is strictly in between $u_L(t)$ and $u_R(t)$. We fix ideas by assuming $u_L > v > u_R$. A similar analysis would apply in the opposite case as well. The only reason why $\tilde{u}$ is not a weak solution of (6.12) is because its jump discontinuities do not travel following the correct Rankine-Hugoniot condition. We have

$$s'(t) = \frac{F(u_R(t)) - F(u_L(t))}{u_R(t) - u_L(t)} \leq \frac{F(v) - F(u_L(t))}{v - u_L(t)}.$$

The speed $s'(t)$ is inherited from the original function $u(t,x)$ (explained in Example 6.2.2), whereas the right hand side of the inequality is the speed that we should have in order to match the Rankine-Hugoniot condition for $\tilde{u}$. The inequality "$\leq$" holds, as opposed to "$>$", as a consequence of the entropy condition (6.21). We observe that the shock travels to the right slower than it should for $\tilde{u}$ to be a solution. Recall that $u_L(t) > u_R(t)$, so the slower speed of the shock to the right means that $\tilde{u}_t + \partial_x F(\tilde{u}) \leq 0$ weakly. This analysis motivates the following definition.

Definition 6.2.4. *We say that a bounded function u is an* entropy solution *of* (6.12) *if it is a weak solution as in Definition 6.2.1 and moreover for all $v \in \mathbb{R}$, the function $\tilde{u}(t,x) = \max(u(t,x), v)$ is a weak subsolution as in Definition 6.2.3.*

Proposition 6.2.5. *The following are equivalent.*

1. *The function u is an entropy solution of* (6.12).
2. *The function u is a weak solution of* (6.12) *and moreover for all $v \in \mathbb{R}$, the function $\tilde{u}(t,x) = \min(u(t,x), v)$ is a weak supersolution in the sense that*

$$\iint (\tilde{u}\varphi_t + F(\tilde{u})\varphi_x) \,\mathrm{d}x \,\mathrm{d}t \leq 0,$$

for every admissible test function φ such that $\varphi \geq 0$.

3. *For all $v \in \mathbb{R}$ and any admissible test function φ such that $\varphi \geq 0$, we have*

$$\begin{aligned} &\iint [u - v]_+ \varphi_t + (F(u) - F(v)) \mathbb{1}_{u>v} \varphi_x \,\mathrm{d}x \,\mathrm{d}t \geq 0, \text{ and} \\ &\iint [v - u]_+ \varphi_t + (F(v) - F(u)) \mathbb{1}_{v>u} \varphi_x \,\mathrm{d}x \,\mathrm{d}t \geq 0. \end{aligned} \tag{6.25}$$

4. *Given any convex function $\eta : \mathbb{R} \to \mathbb{R}$ and any admissible test function $\varphi \geq 0$, let $q : \mathbb{R} \to \mathbb{R}$ be a function such that $q'(u) = \eta'(u) F'(u)$, then*

$$\iint (\eta(u)\varphi_t + q(u)\varphi_x) \,\mathrm{d}x \,\mathrm{d}t \geq 0. \tag{6.26}$$

Proof. We see that (1) is equivalent to (2) because $\min(u(t,x), v) = u(t,x) + v - \max(u(t,x), v)$ and $F(\tilde{u}) = F(u) + F(v) - F(\max(u,v))$. Therefore

$$\iint (\tilde{u}\varphi_t + F(\tilde{u})\varphi_x) \,\mathrm{d}x \,\mathrm{d}t = \iint ([u(t,x) + v - \max(u(t,x), v)]\varphi_t + [F(u) + F(v) - F(\max(u,v)] \,\mathrm{d}x \,\mathrm{d}t,$$

using that both φ_t and φ_x integrate to zero,

$$= \iint ([u(t,x) - \max(u(t,x), v)]\varphi_t + [F(u) - F(\max(u,v)] \,\mathrm{d}x \,\mathrm{d}t,$$

using that u is a weak solution,

$$= \iint ([-\max(u(t,x), v)]\varphi_t + [-F(\max(u,v)] \,\mathrm{d}x \,\mathrm{d}t,$$

using that u is an entropy solution,

$$\leq 0.$$

This shows that (1) $\Rightarrow$ (2). The implication in the opposite direction follows after a similar computation.

We also obtain that (1) $\Leftrightarrow$ (3) with a similar computation noticing that $[u-v]_+ = \max(u,v) - v$ and $(F(u) - F(v))\mathbb{1}_{u>v} = F(\max(u,v)) - F(v)$.

The equivalence with (4) requires a slightly lengthier computation. Since u is a bounded function, let us say that $u \geq -M$ everywhere. We can recover the function η from its second derivative and its values at $-M$ using the following formula

$$\eta(u) = \eta(-M) + \eta'(-M)(u+M) + \int_{-M}^{\infty} \eta''(v)[u-v]_+ \,\mathrm{d}v.$$

It is elementary to check (exercise!) that we also have

$$q(u) = C + \eta'(-M)F(u) + \int_{-M}^{\infty} \eta''(v)[F(u) - F(v)]\mathbb{1}_{u>v} \,\mathrm{d}v.$$

Replacing into the integral of the left hand side of (6.26), using (3) and that $\eta''(v) \geq 0$ for all values of v, we recover the inequality in (6.26). □

The following lemma says that we can replace the constant v in (6.25) for a function v provided that v is also an entropy solution to the same equation (6.12). This lemma is the key to show the uniqueness of entropy solutions to conservation law equations. It is usually presented in more advanced texts in partial differential equations. We give a relatively complete proof here at the expense of some hand-waving regarding limits inside integrals. All these limits can be properly justified using Lebesgue integration theory.

Lemma 6.2.6. *Let u and v be two entropy solutions and $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be a C^1, compactly supported function. Then*

$$\iint [u(t,x) - v(t,x)]_+\varphi_t + (F(u(t,x)) - F(v(t,x)))\mathbb{1}_{u>v}\varphi_x \,\mathrm{d}x \,\mathrm{d}t \geq 0. \tag{6.27}$$

Proof. What makes this proof possible is the doubling variables trick, originally due to Kruzhkov. Let $\eta : \mathbb{R} \to \mathbb{R}$ be a nonnegative smooth function, with integral one, and supported in $[-1,1]$. We scale this function by a parameter $\varepsilon > 0$ by

$$\eta_\varepsilon(x) = \varepsilon^{-1}\eta(\varepsilon^{-1}x).$$

This function η_ε is an approximation of the Dirac delta as $\varepsilon \to 0$. Indeed, we have seen before that for any function g that is continuou at the origin, we have

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}} g(x)\eta_\varepsilon(x) \,\mathrm{d}x = g(0).$$

Furthermore, let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be two integrable functions. Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function that vanishes outside of some bounded set. We would also get

$$\lim_{\varepsilon \to 0} \iint |f(x) - g(y)|\varphi(x,y)\eta_\varepsilon(x-y) \,\mathrm{d}y \,\mathrm{d}x = \int |f(x) - g(x)|\varphi(x,x) \,\mathrm{d}x.$$

This limit follows from the one above when f and g are continuous. The proof is more delicate when we allow the functions f and g to have discontinuities. A complete justification would require the use of Lebesgue integration theory (and specifically, that continuous functions with compact support are dense in L^1). Let us take this fact for granted and carry on with the rest of the proof.

We use the function η_ε to rewrite the double integral (6.27) as a quadruple integral.

$$\begin{aligned}
&\iint [u(t,x) - v(t,x)]_+ \varphi_t + (F(u(t,x)) - F(v(t,x)))\mathbb{1}_{u>v}\varphi_x \, \mathrm{d}x \, \mathrm{d}t \\
&= \lim_{\varepsilon \to 0} \iiiint \left\{ [u(t,x) - v(s,y)]_+ \varphi_t\left(\frac{t+s}{2}, \frac{x+y}{2}\right) + (F(u(t,x)) - F(v(s,y)))\mathbb{1}_{u>v}\varphi_x\left(\frac{t+s}{2}, \frac{x+y}{2}\right) \right\} \\
&\qquad \eta_\varepsilon(x-y)\eta_\varepsilon(t-s) \, \mathrm{d}x \, \mathrm{d}t \, \mathrm{d}y \, \mathrm{d}s.
\end{aligned}$$

The advantage of the quadruple integral in the right hand side is that for every fixed value of (s,y), we can use the value of $v(s,y)$ as the constant in (6.25). Moreover, for each fixed value of (t,x) we can write the definition of entropy solution for $v(s,y)$ using the constant $u(t,x)$ in (6.25). Adding those two inequalities and passing to the limit as $\varepsilon \to 0$, we will recover the result of the lemma.

Let

$$\tilde{\varphi}(t,x,s,y) = \varphi\left(\frac{t+s}{2}, \frac{x+y}{2}\right)\eta_\varepsilon(x-y)\eta_\varepsilon(t-s).$$

This function $\tilde{\varphi}$ will be used as the test function first for (s,y) fixed as a function of (t,x), and later for (t,x) fixed as a function of (s,y).

For every fixed value of (s,y), we use that u is an entropy solution. We apply part (3) in Proposition 6.2.5 with $\varphi(\cdot,\cdot,s,y)$ as the test function. We get

$$\begin{aligned}
0 &\leq \iint [u(t,x) - v(s,y)]_+ \tilde{\varphi}_t(t,x,s,y) + (F(u(t,x)) - F(v(s,y)))\mathbb{1}_{u>v}\tilde{\varphi}_x(t,x,s,y) \, \mathrm{d}x \, \mathrm{d}t, \\
&= \iint [u(t,x) - v(s,y)]_+ \varphi\left(\frac{t+s}{2}, \frac{x+y}{2}\right)\eta_\varepsilon(x-y)\eta'_\varepsilon(t-s) \\
&\quad + [u(t,x) - v(s,y)]_+ \frac{1}{2}\varphi_t\left(\frac{t+s}{2}, \frac{x+y}{2}\right)\eta_\varepsilon(x-y)\eta_\varepsilon(t-s) \\
&\quad + (F(u(t,x)) - F(v(s,y)))\mathbb{1}_{u>v}\varphi\left(\frac{t+s}{2}, \frac{x+y}{2}\right)\eta'_\varepsilon(x-y)\eta_\varepsilon(t-s) \\
&\quad + (F(u(t,x)) - F(v(s,y)))\mathbb{1}_{u>v}\frac{1}{2}\varphi_x\left(\frac{t+s}{2}, \frac{x+y}{2}\right)\eta_\varepsilon(x-y)\eta_\varepsilon(t-s) \, \mathrm{d}x \, \mathrm{d}t.
\end{aligned}$$

Now we fix t and x and use that v is an entropy solution. We use part 3 of Proposition 6.2.5 with the test function $\tilde{\varphi}(t,x,\cdot,\cdot)$, this time as a function of s and y with (t,x) fixed. We get

$$\begin{aligned}
0 &\leq \iint [u(t,x) - v(s,y)]_+ \tilde{\varphi}_s(t,x,s,y) + (F(u(t,x)) - F(v(s,y)))\mathbb{1}_{u>v}\tilde{\varphi}_y(t,x,s,y) \, \mathrm{d}y \, \mathrm{d}s, \\
&= \iint -[u(t,x) - v(s,y)]_+ \varphi\left(\frac{t+s}{2}, \frac{x+y}{2}\right)\eta_\varepsilon(x-y)\eta'_\varepsilon(t-s) \\
&\quad + [u(t,x) - v(s,y)]_+ \frac{1}{2}\varphi_t\left(\frac{t+s}{2}, \frac{x+y}{2}\right)\eta_\varepsilon(x-y)\eta_\varepsilon(t-s) \\
&\quad - (F(u(t,x)) - F(v(s,y)))\mathbb{1}_{u>v}\varphi\left(\frac{t+s}{2}, \frac{x+y}{2}\right)\eta'_\varepsilon(x-y)\eta_\varepsilon(t-s) \\
&\quad + (F(u(t,x)) - F(v(s,y)))\mathbb{1}_{u>v}\frac{1}{2}\varphi_x\left(\frac{t+s}{2}, \frac{x+y}{2}\right)\eta_\varepsilon(x-y)\eta_\varepsilon(t-s) \, \mathrm{d}y \, \mathrm{d}s.
\end{aligned}$$

We integrate the last two displayed inequalities respect to (s, y) and (t, x) respectively. Some terms cancel out and we are left with

$$0 \leq \iiiint [u(t,x) - v(s,y)]_+ \varphi_t \left(\frac{t+s}{2}, \frac{x+y}{2} \right) \eta_\varepsilon(x-y)\eta_\varepsilon(t-s) \\ + (F(u(t,x)) - F(v(s,y))) \mathbb{1}_{u>v} \varphi_x \left(\frac{t+s}{2}, \frac{x+y}{2} \right) \eta_\varepsilon(x-y)\eta_\varepsilon(t-s) \,\mathrm{d}y \,\mathrm{d}s \,\mathrm{d}x \,\mathrm{d}t.$$

The functions η_ε converge to a Dirac delta at zero as $\varepsilon \to 0$. Passing to the limit as $\varepsilon \to 0$, the integral localizes on the diagonal $x = y$ and $t = s$, and we obtain the inequality (6.27). □

When the functions u and v satisfy a mild continuity in time, we can simplify the inequality (6.27) by considering test functions φ that are independent of time. The precise assumption is the following, for any time $t_0 \in [0, \infty)$ and any interval $[a, b] \subset \mathbb{R}$, we must have

$$\lim_{t \to t_0} \int_a^b |u(t_0, x) - u(t,x)| \,\mathrm{d}x = 0 \qquad \text{and} \lim_{t \to t_0} \int_a^b |v(t_0, x) - v(t,x)| \,\mathrm{d}x = 0. \tag{6.28}$$

In terms of functional spaces, the assumption (6.28) is usually written as $u \in C([0,\infty), L^1_{loc}(\mathbb{R}))$. We state the simplified inequality precisely in the following corollary.

Corollary 6.2.7. *Let u and v be two entropy solutions of* (6.12). *Assume that for any time $t_0 \in [0, \infty)$,* (6.28) *holds. Then, for any two times $t_1 > t_0 \geq 0$ and any C^1, compactly supported function $\varphi : \mathbb{R} \to \mathbb{R}$, we have*

$$\int_{\mathbb{R}} ([u(t_1,x) - v(t_1,x)]_+ - [u(t_0,x) - v(t_0,x)]_+) \varphi(x) \,\mathrm{d}x \leq \int_{t_0}^{t_1} \int_{\mathbb{R}} (F(u(t,x)) - F(v(t,x))) \mathbb{1}_{u>v} \varphi'(x) \,\mathrm{d}x \,\mathrm{d}t. \tag{6.29}$$

Sketch proof. Let $\psi_\varepsilon : \mathbb{R} \to \mathbb{R}$ be a family of C^1 functions such that $\psi_\varepsilon(t) = 1$ if $t \in [t_0 + \varepsilon, t_1 - \varepsilon]$, $\psi_\varepsilon(t) = 0$ if $t \notin (t_0, t_1)$. We apply Lemma 6.2.6 using the test function $\psi_\varepsilon(t)\varphi(x)$. The inequality of the corollary follows by taking $\varepsilon \to 0$. The details are left to the reader. □

Theorem 6.2.8. *Let u and v be two entropy solutions of* (6.12). *Assume that there exists a uniform constant C such that for all time $t \geq 0$,*

$$\int_{\mathbb{R}} |u(t,x)| + |v(t,x)| \,\mathrm{d}x \leq C.$$

Assume moreover, that (6.28) *holds. Then the quantity*

$$D(t) = \int_{\mathbb{R}} [u(t,x) - v(t,x)]_+ \,\mathrm{d}x,$$

is monotone decreasing with respect to time.

Proof. Let $\varphi_R(x) = \varphi(x/R)$ where $\varphi(x)$ is a smooth, compactly supported, function such that $0 \leq \varphi \leq 1$ in $\mathbb{R}$, and $\varphi(x) = 1$ if $x \in [-1, 1]$. Note that by construction, $\varphi_R'(x) = \varphi'(x/R)/R \leq C/R$ for some constant C.

We use φ_R as the test function in (6.29) and pass to the limit as $R \to \infty$. We obtain

$$\begin{aligned} D(t_1) - D(t_0) &\leq \lim_{R \to \infty} \int_{t_0}^{t_1} \int_{\mathbb{R}} (F(u(t,x)) - F(v(t,x))) \mathbb{1}_{u>v} \varphi_R'(x) \,\mathrm{d}x \,\mathrm{d}t, \\ &\leq \lim_{R \to \infty} \frac{C}{R} \max |F'| \left(\int |u(t,x) - v(t,x)| \,\mathrm{d}x \right) = 0. \end{aligned}$$

Note that the maximum of F' must be evaluated only in the range of the functions u and v, that are bounded. Moreover, since $|u - v| < |u| + |v|$, the integral on the right hand side if also bounded by a constant C.

We concluded that $D(t_1) \leq D(t_0)$ for any $t_1 > t_0 \geq 0$, which means that D is monotone decreasing. □

The uniqueness of solutions is a consequence of Theorem 6.2.8.

Corollary 6.2.9. *There exists at most one entropy solution of the initial value problem*

$$\begin{aligned} u_t + \partial_x F(u) &= 0, \\ u(0,x) &= u_0(x), \end{aligned}$$

such that (6.28) *holds and* $\int |u(t,x)| \,\mathrm{d}x$ *is bounded uniformly in time.*

Proof. If there were two such solutions u and v, because of Theorem 6.2.8, we would have that the two quantities

$$\int_{\mathbb{R}} [u(t,x) - v(t,x)]_+ \,\mathrm{d}x \qquad \text{and} \qquad \int_{\mathbb{R}} [v(t,x) - u(t,x)]_+ \,\mathrm{d}x$$

are monotone decreasing in time. Since these quantities are nonnegative and initially zero, they must be zero for all time and therefore $u \equiv v$. □

Remark 6.2.10. In mathematics, it is customary to use the term *weak solutions* to refer to solutions in the sense of Definition 6.2.1. This notion of solution applies to other equations besides conservation laws. We may verify that a (possibly non-smooth) function u solves the wave equation in $\mathbb{R} \times \mathbb{R}^d$ if for all φ compactly supported in $\mathbb{R} \times \mathbb{R}^d$ we have the identity

$$\iint_{\mathbb{R}\times\mathbb{R}^d} u(\varphi_{tt} - \Delta\varphi) \,\mathrm{d}x \,\mathrm{d}t = 0.$$

This notion of solution makes a lot of sense for linear (and perhaps semilinear) equations. However, as we saw above, it has severe pitfalls when we try to apply it to nonlinear equations. Personally, I dislike the term *weak solution* because it makes us believe that this definition gives us a notion of solution that actually has some significance. Given how flawed this definition turns out to be, I would rather not give it the status of *solution*.

6.3 The Hamilton-Jacobi equation

We consider the Hamilton-Jacobi equation.

$$u_t + H(t,x,u,\nabla u) = 0. \tag{6.30}$$

We discussed in Section 6.1.3 how to solve the equation (6.30) for a short period of time using the method of characteristics. Like in most non-linear first order equations, the characteristic curves may cross. This means that there cannot exist a solution, in the classical sense, for all positive time. Like we did for conservation law equations, we need to develop a generalized notion of solution that allows us to continue the equation past the time when characteristic curves collide. We need to understand how to properly verify the equation for functions that are not necessarily differentiable everywhere.

When H is convex, the equation represents the minimum cost in certain optimal control problem. We describe this problem below and we use it to derive a formula for the solution. This formula makes sense for all $t > 0$, but it gives us a solution whose derivatives contains discontinuities.

In Section 6.2.2, we discussed weak solutions of conservation law equations. The idea was to *integrate by parts* the derivatives from the equation into an arbitrary test function. We cannot do the same in the context of the Hamilton-Jacobi equations, since the derivatives are inside a nonlinear function H. It turns out that the appropriate generalized solution is known by the name of *viscosity solution*, even though there is no connection between this equation and viscous fluids.

6.3.1 An optimal control problem.

Given a function $u_0 : \mathbb{R}^d \to \mathbb{R}$, let us consider the following minimization problem

$$u(t,x) = \min\left\{ u_0(\gamma(0)) + \int_0^t L(\gamma(s), \gamma'(s)) \,\mathrm{d}s : \text{ from all curves } \gamma : [0,t] \to \mathbb{R}^n \text{ with } \gamma(t) = x \right\}. \tag{6.31}$$

The expression above represents an optimal control problem. We can think of the function $L(x,v)$ as a cost for the curve $\gamma(\cdot)$ to move at velocity v through the point x. The function u_0 represents a terminal cost depending on the final value of the curve $\gamma(t)$. The purpose of this minimization problem would be to find the curve that achieves the minimum total cost and determine in that way the function $u(t,x)$.

The minimization described in (6.31) is taken among all continuous paths γ that are piecewise C^1. Using calculus of variations, it is possible to prove that when L and u_0 are continuous, u_0 is bounded, and $L(x,v) \to +\infty$ as $|v| \to \infty$, there exists a path γ achieving this minimum. Moreover, assuming that $L(x,v)$ is smooth and strictly convex with respect to v, the minimization path is everywhere differentiable. The minimization path γ may not be unique.

Lemma 6.3.1. *Let γ_0 be a curve that achieves the minimum in* (6.31)*. Then, for any $t_0 \in (0,t)$, we have*

$$u(t,x) = u(t_0, \gamma_0(t_0)) + \int_{t_0}^{t} L(\gamma_0(s), \gamma_0'(s)) \, \mathrm{d}s.$$

Proof. Let $\gamma_1 : [0,t_0] \to \mathbb{R}^d$ be the curve that achieves the minimum in (6.31) corresponding to the value of $u(t_0, \gamma_0(t_0))$. That is

$$u(t_0, \gamma_0(t_0)) = u_0(\gamma_1(0)) + \int_0^{t_0} L(\gamma_1(s), \gamma_1'(s)) \, \mathrm{d}s.$$

We intend to prove that γ_0 achieves the minimum as well.

The curve γ_0 restricted to $[0,t_0]$ in another candidate curve for the same minimization problem. Therefore

$$u(t_0, \gamma_0(t_0)) = u_0(\gamma_1(0)) + \int_0^{t_0} L(\gamma_1(s), \gamma_1'(s)) \, \mathrm{d}s \leq u_0(\gamma_0(0)) + \int_0^{t_0} L(\gamma_0(s), \gamma_0'(s)) \, \mathrm{d}s. \tag{6.32}$$

Let us build a curve $\tilde{\gamma}_0 : [0,t] \to \mathbb{R}^d$ by gluing the curve γ_1 with the curve γ_0 restricted to $[t_0,t]$.

$$\tilde{\gamma}_0(s) := \begin{cases} \gamma_1(s) & \text{for } s \in [0,t_0], \\ \gamma_0(s) & \text{for } s \in [t_0,t]. \end{cases}$$

This curve $\tilde{\gamma}_0$ is a candidate for the minimization problem (6.31) corresponding to $u(t,x)$. Therefore

$$u(t,x) = u_0(\gamma_0(0)) + \int_0^{t} L(\gamma_0(s), \gamma_0'(s)) \, \mathrm{d}s \leq u_0(\tilde{\gamma}_0(0)) + \int_0^{t} L(\tilde{\gamma}_0(s), \tilde{\gamma}_0'(s)) \, \mathrm{d}s.$$

Since the curves coincide in the interval $[t_0,t]$, we deduce that

$$u_0(\gamma_0(0)) + \int_0^{t_0} L(\gamma_0(s), \gamma_0'(s)) \, \mathrm{d}s \leq u_0(\gamma_1(0)) + \int_0^{t_0} L(\gamma_1(s), \gamma_1'(s)) \, \mathrm{d}s.$$

This is the opposite inequality as (6.32). Therefore, the equality holds and the lemma follows. ☐

The following lemma is sometimes referred to as the dynamic programming principle.

Lemma 6.3.2. *We have the following identity for any $h \in (0,t)$,*

$$u(t,x) = \min \left\{ u(t-h, \gamma(0)) + \int_0^{h} L(\gamma(s), \gamma'(s)) \, \mathrm{d}s : \textit{ for all curves } \gamma : [0,h] \to \mathbb{R}^n \textit{ with } \gamma(h) = x \right\}. \tag{6.33}$$

Proof. Let γ_0 be the curve that achieves the minimum in (6.31) and γ_1 be the curve that achieves the minimum on the right hand side of (6.33).

The fact that γ_0 is the curve that achieves the minimum in (6.31) means that $\gamma_0(t) = x$ and

$$u(t,x) = u_0(\gamma_0(0)) + \int_0^{t} L(\gamma_0(s), \gamma_0'(s)) \, \mathrm{d}s. \tag{6.34}$$

Conversely, since γ_1 achieves the minimum on the right hand side of (6.33), it must give us a smaller value than any other candidate curve. In particular, we may compare it with $\tilde{\gamma}_1(s) := \gamma_0(t-h+s)$ and get

$$u(t-h,\gamma_1(0)) + \int_0^h L(\gamma_1(s),\gamma_1'(s)) \leq u(t-h,\gamma_0(t-h)) + \int_{t-h}^t L(\gamma_0(s),\gamma_0'(s))\,\mathrm{d}s = u(t,x). \tag{6.35}$$

We used Lemma 6.3.1 for the last equality.

Let $\gamma_2 : [0,t-h] \to \mathbb{R}^n$ be the curve the achieves the minimum for the expression (6.31) corresponding to $u(t-h,\gamma_1(0))$. That is

$$u(t-h,\gamma_1(0)) = u_0(\gamma_2(0)) + \int_0^{t-h} L(\gamma_2(s),\gamma_2'(s))\,\mathrm{d}s. \tag{6.36}$$

We may join the two curves γ_2 and γ_1 and build another candidate curve for the minimum of (6.31). Let

$$\gamma_3(s) := \begin{cases} \gamma_2(s) & \text{for } s \in [0,t-h], \\ \gamma_1(s-t+h) & \text{for } s \in [t-h,t]. \end{cases}$$

Since the curve γ_0 achieves the minimum in (6.31), we have

$$u(t,x) \leq u(0,\gamma_3(0)) + \int_0^t L(\gamma_3(s),\gamma_3'(s))\,\mathrm{d}s$$

Using (6.36)

$$\begin{aligned} &= u(t-h,\gamma_1(0)) + \int_{t-h}^t L(\gamma_3(s),\gamma_3'(s))\,\mathrm{d}s \\ &= u(t-h,\gamma_1(0)) + \int_0^h L(\gamma_1(s),\gamma_1'(s))\,\mathrm{d}s. \end{aligned}$$

Because of the definition of u as a minimum, we can restrict the curve γ to $[0,t-h]$ and get

$$u(t-h,\gamma(t-h)) \leq u_0(\gamma(0)) + \int_0^{t-h} L(\gamma(s),\gamma'(s))\,\mathrm{d}s.$$

This is the opposite inequality as (6.35). Therefore, equality holds and we finish the proof. □

We will use Lemma 6.3.2 to derive an equation for u. This derivation works under the assumption that u is a differentiable function and the minimization problem is achieved by a unique differentiable curve $\gamma(\cdot)$. Indeed, for h sufficiently small, we would have

$$u(t-h,\gamma(0)) = u(t,x) - hu_t(t,x) - h\gamma'(h)\cdot\nabla u(t,x) + o(h),$$
$$\int_0^h L(\gamma(s),\gamma'(s))\,\mathrm{d}s = hL(x,\gamma'(h)) + o(h)$$

If we call $v = \gamma'(h)$, the minimization problem (6.33) becomes

$$u(t,x) = \min\{u(t,x) + h\left(-u_t(t,x) - v\cdot\nabla u(t,x) + L(x,v)\right) : v \in \mathbb{R}^n\} + o(h).$$

Thus, by passing to the limit as $h \to 0$, we deduce that

$$u_t = \min_v \left(-v\cdot\nabla u(t,x) + L(x,v)\right).$$

Let us define the function H as the Legendre transform of L, that is

$$H(x,p) = \max_v \left(v\cdot p - L(x,v)\right).$$

We deduced that the function u defined in (6.31) must satisfy the Hamilton-Jacobi equation

$$u_t + H(x, \nabla u) = 0.$$

Note that the curve achieving the minimum in (6.31) would correspond to the characteristic curve though the point (t, x). The method of characteristics only gives us a solution of the Hamilton-Jacobi equation for small values of t. For large values of t, it may happen that several characteristics cross at the point (t, x). This corresponds to the lack of uniqueness in the minimization problem 6.31. In fact, the function u defined in (6.31) may not be differentiable for large values of t, depending on the initial data u_0. It should be appropriate to say that the function u is the *right* solution to the Hamilton-Jacobi equation nonetheless. After all, the optimal control problem (6.31) was the start of our analysis, from which we motivated the Hamilton-Jacobi equation.

6.3.2 Viscosity solutions.

The right notion of solution for the Hamilton-Jacobi equation, which we discuss below, is called *viscosity solution*. Despite its misleading name, there is no connection between this and viscous fluids. We give the definition right away. We assume that H is a continous function with respect to all its variables and $\Omega \subset \mathbb{R}^n$ is an open set.

Definition 6.3.3. *We say a continuous function $u : [0, T] \times \Omega \to \mathbb{R}$ is a* **subsolution** *(alt* **supersolution***) of the equation* (6.30) *if every time there exists a C^1 function φ, $r > 0$ and a point $(t_0, x_0) \in (0, T] \times \Omega$, such that*

$$\begin{aligned} \varphi(t_0, x_0) &= u(t_0, x_0), \\ \varphi(t, x) &\geq u(t, x) \quad (\text{alt} \leq) \text{ for all } t \in (t_0 - r, t_0] \times B_r(x_0), \end{aligned}$$

then $\varphi_t + H(t, x, \varphi, \nabla\varphi) \leq 0$ (alt ≥ 0).

A continuous function u is a **viscosity solution** *in $(0, T] \times \Omega$ when it is both a subsolution and a supersolution.*

Compared with the notion of weak solutions for conservation laws, this definition uses a different mechanism to translate the differentiability requirement from the function u to the test function φ.

If u is a C^1 function, we see that whenever there is a smooth function φ touching u from above at the point (t_0, x_0) as in Definition 6.3.3, then $\varphi_t(t_0, x_0) \leq u_t(t_0, x_0)$ and $\nabla\varphi(t_0, x_0) = \nabla u(t_0, x_0)$, therefore $\varphi_t + H(t_0, x_0, \varphi, \nabla\varphi) \leq u_t + H(t_0, x_0, u, \nabla u)$ at (t_0, x_0). This is the motivation of the definition. Moreover, it is also the justification of the following proposition.

Proposition 6.3.4. *Let $u : [0, T] \times \Omega \to \mathbb{R}$ be differentiable in $(0, T] \times \Omega \to \mathbb{R}$. Then u is a subsolution (alt supersolution) of* (6.30) *if and only if $u_t + H(t, x, u, \nabla) \leq 0$ (alt $\geq$) for all points $(t, x) \in (0, T] \times \Omega$.*

Note that the notion of viscosity solution imposes a condition on u only at the points where there exists some smooth function φ which is tangent from one side. There may be some points (t_0, x_0) where such a function φ does not exist. Nothing needs to be checked at those points according to Definition 6.3.3.

Here, we give a definition of viscosity solutions that is made for time-dependent first order equations. The definition can also be applied to time-independent equations. In that case, we would just ignore the time variable and think of functions $u(x)$ and $\varphi(x)$ that are constant in t. Equivalently, we can use the definition above to test the notion of stationary solutions of the Hamilton-Jacobi equations.

The definition can easily be extended to second order (parabolic) equations of the form

$$u_t - F(t, x, u, \nabla u, D^2 u) = 0.$$

In that case, we have to impose the monotonicity condition saying that $F(t, x, u, p, X) \geq F(t, x, u, p, Y)$ whenever $X \geq Y$ (meaning that $X - Y$ is a positive definite matrix in $R^{n \times n}$).

The definition does not change if we require $\varphi(t, x) > u(t, x)$ for all $(t, x) \in (t_0 - r, t_0] \times B_r(x_0) \setminus (t_0, x_0)$. Indeed, for any test function φ as in Definition 6.3.3, we can consider $\tilde{\varphi}(t, x) = \varphi(t, x) + |t - t_0|^2 + |x - x_0|^4$ so that the derivatives of $\tilde{\varphi}$ and φ coincide at (t_0, x_0), and $\tilde{\varphi} > \varphi \geq u$ at all neighboring points.

Examples

The following examples can be verified using Definition 6.3.3.

1. Given the equation $u_t - |u_x| = 0$, the function
$$u(t,x) = \begin{cases} t-x & \text{when } x \in (0,t), \\ t+x & \text{when } x \in (-t,0), \\ 0 & \text{otherwise,} \end{cases}$$
is a viscosity supersolution but not a viscosity subsolution. It fails the subsolution criteria only at the top point $x = 0$.

2. The function $u(t,x) = 1 - |x|$ is a stationary solution of the equation $u_t + |u_x|^2 - 1 = 0$. The function $u(t,x) = |x| - 1$ is a subsolution of the same equation, but not a supersolution.

3. Considering the notion of viscosity solutions for time independent problems, the example above turns into the following strange (but correct) situation. The function $u(x) = 1-|x|$ is the viscosity solution to the equation $|u_x|^2 = 1$ in $(-1,1)$. The function $u(x) = |x| - 1$ is a subsolution but not a supersolution. If we considered the equation $-|\nabla u|^2 = -1$, the choice of these functions would be reversed.

Remark 6.3.5. It makes sense to extend the notion of viscosity sub- and super-solutions to semicontinuous functions. When it comes to sub-solutions, Definition 6.3.3 makes sense for an upper semi-continuous function u. Conversely, it makes sense to extend the notion of viscosity super-solution given in Definition 6.3.3 to lower semi-continuous functions.

6.4 Exercises

Exercise 6.1. *Consider the transport equation*
$$u_t + a(t,x) \cdot \nabla u = 0.$$
Prove that if u is a solution and $\varphi : \mathbb{R} \to \mathbb{R}$ is an arbitrary C^1 function, then $\varphi(u(t,x))$ is also a solution.

Exercise 6.2. *Let $u : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ be the unique solution of the first order equation*
$$u_t + b(x) \cdot \nabla u = 0,$$
with initial data $u(0,x) = u_0(x)$. We assume that $b(x)$ is globally bounded and Lipchitz with respect to x, so that the equation is well posed. Let us also assume that u_0 is compactly supported.

(a) Prove that for all $t > 0$, the function $u(t,\cdot)$ is compactly supported in $\mathbb{R}^d$.

(b) Prove that $\operatorname{div} b(x) = 0$ if and only if the integral of u is constant in time for all $u_0 : \mathbb{R}^d \to \mathbb{R}$ that is compactly supported. In other words, $\operatorname{div} b(x) = 0$ if and only if for all initial data u_0,
$$\int_{\mathbb{R}^d} u(t,x)\, \mathrm{d}x \text{ is constant with respect to } t.$$

Exercise 6.3. *We look for a function $u : [0,T) \times \mathbb{R} \to \mathbb{R}$ solving Burgers equation*
$$u_t + u u_x = 0,$$
with a prescribed initial value $u(0,x) = u_0(x)$.

(a) Prove that the characteristic curves do not cross for
$$t < \frac{1}{\sup_x (-u_0'(x))_+}.$$

(b) Prove that a classical solution u exists for all positive time if and only if the initial data u_0 is monotone increasing: $u_0(x) \geq u_0(y)$ whenever $x > y$.

(c) If $u_0(x) = \sin(x)$, find the maximal time of existence of a classical solution.

Exercise 6.4. *Let $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a C^1 function solving Burgers equation for all times $t \in \mathbb{R}$ and $x \in \mathbb{R}$ (it is called an eternal solution). Prove that u is constant.*

Exercise 6.5. *Let u be a smooth solution to a first order equation*

$$u_t + b(t,x,u) \cdot \nabla u = c(t,x,u).$$

Assume that b is bounded and that $c(t,x,u) \leq C_0|u|$. Prove that

$$u(t,x) \leq e^{C_0 t} \max(0, \sup_x u_0(x)).$$

Exercise 6.6. *Consider two smooth functions u_1 and u_2 that solve the same first order equation*

$$u_t + b(t,x,u) \cdot \nabla u(t,x) = c(t,x,u) \qquad \text{in } [0,T) \times \mathbb{R}. \tag{6.37}$$

Let $x_1 < x_2$ be two points so that $u_1(0,x_1) = u_2(0,x_1)$, $u_1(0,x_2) = u_2(0,x_2)$ and $u_1(0,x) > u_2(0,x)$ for all $x \in (x_1,x_2)$. Prove that there are two curves $\gamma_1, \gamma_2 : [0,T) \to \mathbb{R}$ so that

$$\begin{aligned}
\gamma_1(0) &= x_1, \\
\gamma_2(0) &= x_2, \\
u_1(t,\gamma_1(t)) &= u_2(t,\gamma_1(t)), \\
u_1(t,\gamma_2(t)) &= u_2(t,\gamma_2(t)), \\
u_1(t,x) &> u_2(t,x) \qquad \text{for all } x \in (\gamma_1(t),\gamma_2(t)).
\end{aligned}$$

Exercise 6.7. *Let u be a solution of the Hamilton-Jacobi equation*

$$u_t + u_x^2/2 = 0.$$

Prove that its derivative, $v = u_x$, is a solution of the Burgers equation

$$v_t + v v_x = 0.$$

Exercise 6.8. *Consider the Hamilton-Jacobi equation*

$$u_t + u_x^2 = 0,$$

with prescribed initial value $u(0,x) = u_0(x)$.

Prove that if u_0 is smooth and convex, then a smooth solution $u(t,x)$ exists for all positive time (i.e. the characteristic curves never cross).

Exercise 6.9. *Let u be a solution to the Burgers equation*

$$u_t + u u_x = 0, \quad \text{for } x \in \mathbb{R}, t > 0,$$

with initial data $u(0,x) = u_0$.

(a). Assume that u_0 is compactly supported. Prove that $u(t,\cdot)$ is compactly supported for all $t > 0$ as long as the solution exists.

(b). Assume that u_0 is compactly supported. Prove that the integral

$$I(t) = \int_{\mathbb{R}} u(t,x)^2 \, \mathrm{d}x,$$

is constant in t for as long as the smooth solution exists.

(c). For all $x \in \mathbb{R}$ and $t > 0$, $u_x(t,x) \leq 1/t$. This estimate is independent of the initial data u_0 for as long as the smooth solution exists.

Hint. *Differentiate the equation and derive a transport equation for u_x. Recall that the solution to the ODE $P' = -P^2$ with $P(0) = +\infty$ is $P(t) = 1/t$.*

(d). Using (b) and (c), prove the a priori estimate

$$\max_{x\in\mathbb{R}} u(t,x) \leq \left(\frac{3\int u_0(x)^2\, \mathrm{d}x}{t}\right)^{1/3}.$$

Note. *It is not hard to check that $\max_{x\in\mathbb{R}} u(t,x)$ is constant in time for as long as the smooth solution exists. The estimate from (d) gives us a very indirect proof that the smooth solution may not last forever.*

Note. *The results in parts (a), (c) and (d) are remain true for entropy solutions for all $t > 0$. In this question, you are only asked to verify them for classical solutions.*

Exercise 6.10. *Let $H : [0,\infty) \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be given. Assume that $u : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ and $p : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ are C^2 functions that solve the system*

$$u_t + \frac{\partial H}{\partial p} \cdot \nabla u = -H(t,x,u,p) + \frac{\partial H}{\partial p} \cdot p,$$
$$p_t + \frac{\partial H}{\partial p} \cdot \nabla p = -\frac{\partial H}{\partial x} - \frac{\partial H}{\partial u} p.$$

Assuming that $\nabla u(0,x) = p(0,x)$, prove that $\nabla u(t,x) = p(t,x)$ for all $t > 0$.

Exercise 6.11. *Let $u(t,x)$ be the solution of the Hamilton-Jacobi equation*

$$u_t = H(\nabla u).$$

Assume that $H(p) = \min_v p \cdot v + L(v)$, where L is some convex function. Let $X(\cdot)$ be the characteristic curve passing through the point (t,x). Prove that

$$u(t,x) = u(0, X(0)) + \int_0^t L(-X'(s))\, \mathrm{d}s.$$

Exercise$*$ 6.12. *Let $u : \mathbb{R}^n \times [0,T) \to \mathbb{R}$ be a function which solves the equation*

$$u_t = |\nabla u|^2,$$

with initial data $u(0,x) = u_0(x)$. Prove that for any $t \in (0,T)$,

$$u(t,x) = \max\left\{u_0(y) - \frac{|x-y|^2}{4t} : y \in \mathbb{R}^n\right\}.$$

Exercise$*$ 6.13. *Let $u : \mathbb{R}^n \times [0,T) \to \mathbb{R}$ be a function which solves the equation*

$$u_t = |\nabla u|,$$

with initial data $u(0,x) = u_0(x)$. Prove that for any $t \in (0,T)$,

$$u(t,x) = \max\{u_0(y) : |x-y| \leq t\}.$$

Exercise$*$ 6.14. *For $\varepsilon > 0$, let $u : \mathbb{R}^n \times [0,T) \to \mathbb{R}$ be a function which solves the equation*

$$u_t = |\nabla u|^2 + \varepsilon \Delta u,$$

with initial data $u(0,x) = u_0(x)$ which is assumed to be bounded and continuous.

(a). Prove that the function $v(t,x) = \frac{1}{\varepsilon} u\left(\frac{t}{\varepsilon}, x\right)$ *solves*

$$v_t = |\nabla v|^2 + \Delta v.$$

(b). Prove that the function $w(t,x) = e^{v(t,x)}$ *solves*

$$w_t = \Delta w.$$

(c) Using the heat kernel, write down an explicit formula for $u(t,x)$ *in terms of* u_0*,* x*,* t *and* ε*.*

(d). Prove that

$$\lim_{\varepsilon \to 0} u(t,x) = \max\left\{ u_0(y) - \frac{|x-y|^2}{4t} : y \in \mathbb{R}^n \right\}.$$

Exercise 6.15. *Verify that the function*

$$u(t,x) = \begin{cases} 1 & \text{if } 0 < x < t/2, \\ -1 & \text{if } -t/2 < x < 0, \\ 0 & \text{otherwise,} \end{cases}$$

is a weak solution of the Burgers equation $u_t + u\,u_x = 0$*, however, it is not an entropy solution.*

Exercise∗ 6.16. *Give an example of a non-identically-zero bounded function* $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *such that* $u(t,x) = 0$ *if* $t \le 0$ *or* $t \ge 1$*, and* u *is a weak solution of the Burgers equation everywhere (but not an entropy solution).*

Hint. Use the example from the previous question in a periodic arrangement so that the function u flows from being identically zero at $t = 0$ to a periodic step function that alternates values ± 1 in intervals of length one at $t = 2$. Then use a similar construction to flow back to zero. This construction gives you a weak solution so that $u(t,x) = 0$ whenever $t < 0$ or $t > 4$. Scale it to obtain the one requested in the question.

Exercise 6.17. *Describe explicitly the unique entropy solution to the Burgers equation* $u_t + u\,u_x = 0$ *with initial data*

$$u(0,x) = \begin{cases} 1 & \text{if } x \in [0,1], \\ 0 & \text{otherwise.} \end{cases}$$

Hint. The function $u(t,x)$ is equal to 1 in some region, to x/t in another, and to 0 elsewhere. You need to determine the curves separating these domains using the Rankine-Hugoniot condition.

Exercise 6.18. *Let* $u : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ *be a bounded integral function that is a weak solution to the linear transport equation* $u_t + u_x = 0$*. Assume* $u \in C([0,\infty), L^1_{loc})$*, in the sense that the continuity condition in (6.28) holds. Prove that* $u(t,x) = u(0, x-t)$*.*

Exercise∗ 6.19. *Let* u *and* v *be two entropy solutions of the conservation law equation*

$$u_t + \partial_x f(u) = v_t + \partial_x f(v) = 0.$$

Assume that $u, v \in C([0,\infty), L^1_{loc}(\mathbb{R}))$*, in the sense that the equation (18) in the notes holds. Assume that both bounded functions* u *and* v *take values in an interval* $I \subset \mathbb{R}$ *and let* $M = \max\{f'(w) : w \in I\}$*. Prove that for any* $a < b$*, the quantity*

$$D(t) = \int_{a+Mt}^{b-Mt} [u(t,x) - v(t,x)]_+ \,\mathrm{d}x,$$

is monotone decreasing in t*.*

Conclude that the initial value problem

$$u_t + \partial_x f(u) = 0,$$
$$u(0,x) = u_0(x)$$

has the following finite speed of propagation property. The values of $u_0(x)$ *for* $x \in [x_0 - Mt_0, x_0 + Mt_0]$ *determine the values of* $u(t,y)$ *for all* (t,y) *such that* $t \in [0,t_0]$ *and* $|y - x_0| < M(t_0 - t)$*.*

Exercise 6.20. *Let $u, v \in C([0,\infty), L^1_{loc}(\mathbb{R}))$ be two entropy solutions to the same conservation law equation. Prove that if $u(0,x) \leq v(0,x)$ for all $x \in \mathbb{R}$, then $u(t,x) \leq v(t,x)$ for (almost) all $t > 0$ and $x \in \mathbb{R}$.*

Exercise 6.21. *Let u and v be two entropy solutions to the same conservation law equation. Prove that $\max(u,v)$ is a weak subsolution of the same equation.*

Exercise 6.22. *Let u_k be a uniformly bounded sequence of entropy solutions of $\partial_t u_k + \partial_x f(u) = 0$. Assume that $u_k \to u$ in $L^1_{loc}([0,\infty) \times \mathbb{R})$ in the sense that for any compact subset $K \subset [0,\infty) \times \mathbb{R}$ we have*

$$\lim_{k\to\infty} \iint_K |u_k(t,x) - u(t,x)| \, \mathrm{d}x \, \mathrm{d}t = 0.$$

Prove that u is also an entropy solution of the same equation.

Exercise 6.23. *Let $A : \mathbb{R} \to \mathbb{R}$ be an arbitrary smooth function. Let u be a **smooth** compactly-supported solution of the conservation law equation*

$$u_t + \partial_x(A(u)) = 0.$$

Prove that $\max\{u(t,x) : x \in \mathbb{R}\}$ is constant in time.

Assume that u is a compactly-supported entropy solution. Prove that $\operatorname{esssup}_{x\in\mathbb{R}} u(t,x)$ is monotone decreasing in time. Show an example in which it is strictly decreasing.

Note. Here, $\operatorname{esssup} u(t,x)$ denotes the essential suppremum of the function u for $x \in \mathbb{R}$. If u is piecewise C^1 and we make it equal to the maximum between u_L and u_R at every jump discontinuity, then $\operatorname{esssup} u(t,x) = \max u(t,x)$.

Exercise 6.24. *Prove that the maximum of two viscosity subsolutions of the Hamilton-Jacobi equation is also a subsolution.*

Exercise 6.25. *Let $u(t,x)$ be given by the following function*

$$u(t,x) = \begin{cases} t - x & \text{if } x \in (0,t), \\ t + x & \text{if } x \in (-t,0), \\ 0 & \text{elsewhere.} \end{cases}$$

Verify that u solves the Hamilton-Jacobi equation $u_t + u_x^2 = 0$ except on three lines. Note that $u(0,x) = 0$. Prove that it is not a viscosity solution in $(0,\infty) \times \mathbb{R}$.

Exercise 6.26. *Let u be a viscosity solution of the Hamilton-Jacobi equation in $[0,\infty) \times \mathbb{R}^d$. Assume that u is differentiable at the single point (t_0,x_0). Prove that the equation holds pointwise at that point.*

$$u_t(t_0,x_0) + H(\nabla u(t_0,x_0)) = 0.$$

Exercise 6.27. *Let $u : [0,T] \times \Omega \to \mathbb{R}$ be a continuous function such that every time there exists a C^1 function φ, $r > 0$ and a point $(t_0,x_0) \in (0,T) \times \Omega$, such that*

$$\begin{aligned} &\varphi(t_0,x_0) = u(t_0,x_0), \\ &\quad \varphi(t,x) \geq u(t,x) \quad \text{for all } t \in (t_0 - r, t_0 + r) \times B_r(x_0) \subset [0,T] \times \Omega, \end{aligned}$$

then $\varphi_t + H(t,x,\varphi,\nabla\varphi) \leq 0$.

This is perhaps the most standard definition of viscosity subsolution, but it is not the one we are using. In Definition 6.3.3, we only require that $\varphi \geq u$ for the previous values of time: $(t_0 - r, t_0] \times B_r(x_0)$. The objective of this question is to prove that both notions are equivalent.

(a) *Prove that if u is a subsolution in $(0,T] \times \Omega$ as in Definition 6.3.3. Then it satisfies the property described above.*

(b) Let φ be a test function as in Definition 6.3.3. Prove that there exists a nonnegative number b such that $\tilde{\varphi}(t,x) = \varphi(t,x) + (b-\varepsilon)(t-t_0)$ is another valid test function, but $\tilde{\varphi}(t,x) = \varphi(t,x) + (b+\varepsilon)(t-t_0)$ is not for any $\varepsilon > 0$.

(c) Let φ as in (b). Prove that there is a sequence of points $x_k \to x_0$, $t_k \to t_0$, with $t_k < t_0$, and $h_k \geq 0$ so that the function

$$\varphi_k(t,x) = \varphi(t,x) + (b + 1/k)(t - t_0) + |t - t_0|^2 + |x - x_0|^2 + h_k,$$

touches u from above at the point (t_k, x_k) in the sense described above (in the interval $(t_k - r_k, t_k + r_k)$ for some $r_k > 0$).

(d) Conclude that the condition described in this question is equivalent to the notion of subsolution given in Definition 6.3.3.

Exercise$*$ 6.28. *Let u_k be a sequence of subsolutions of the Hamilton-Jacobi equation $\partial_t u_k + H(\nabla u_k) = 0$. Assume that $u_k \to u$ uniformly in $[0,T] \times \Omega$. Prove that u is also a subsolution to the same equation.*

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