# Conservation law equations : problem set

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## 1 Method of characteristics

For the problems in this section, assume that the solutions are all classical.

**Question 1** *Let $u$ be a solution to a first order equation*

$$u_t + b(t,x,u) \cdot \nabla u = c(t,x,u).$$

*Assume that $c(t,x,u) \leq C_0|u|$. Prove that*

$$u(t,x) \leq e^{C_0 t} \max(0, \max_x u_0(x)).$$

**Question 2** *Let $u$ be a solution of the Hamilton-Jacobi equation*

$$u_t + u_x^2/2 = 0.$$

*Prove that its derivative, $v = u_x$, is a solution of the Burgers equation*

$$v_t + v v_x = 0.$$

**Question 3** *Let $u$ be a solution to the Burgers equation*

$$u_t + u u_x = 0, \quad \text{for } x \in \mathbb{R}, t > 0,$$

*with initial data $u(x,0) = u_0$.*

*(a). Assume that $u_0$ is compactly supported. Prove that $u(\cdot, t)$ is compactly supported for all $t > 0$.*

*(b). Assume that $u_0$ is compactly supported. Prove that the integral*

$$I(t) = \int_{\mathbb{R}} u(x,t)^2 \, \mathrm{d}x,$$

*is constant in $t$ for as long as the smooth solution exists.*

*(c). For all $x \in \mathbb{R}$ and $t > 0$, $u_x(x,t) \leq 1/t$. This estimate is independent of the initial data $u_0$.*

***Hint.*** *Differentiate the equation and derive a transport equation for $u_x$. Recall that the solution to the ODE $\dot{P} = -P^2$ with $P(0) = +\infty$ is $P(t) = 1/t$.*

*(d). Using (b) and (c), prove the a priori estimate*

$$\max_{x \in \mathbb{R}} u(x,t) \leq \left( \frac{3 \int u_0(x)^2 \, \mathrm{d}x}{t} \right)^{1/3}.$$

**Note.** *It is not hard to check that $\max_{x \in \mathbb{R}} u(x,t)$ is constant in time for as long as the smooth solution exists. The estimate from (d) gives us a very indirect proof that the smooth solution may not last forever.*

**Question 4** *We look for a function $u : \mathbb{R} \times (a,b) \to \mathbb{R}$ solving Burgers equation*

$$u_t + u u_x = 0 \qquad \text{with } u(0,x) = u_0(x).$$

*Prove that if $u_0$ is smooth and $u_0' \geq 0$, then the characteristic curves never cross and the classical solution exists for all $t > 0$.*

## 2 Weak solutions

**Question 5** *Verify that the function*

$$u(t,x) = \begin{cases} 1 & \text{if } 0 < x < t/2, \\ -1 & \text{if } -t/2 < x < 0, \\ 0 & \text{otherwise,} \end{cases}$$

*is a weak solution of the Burgers equation $u_t + u\,u_x = 0$, however, it is not an entropy solution.*

**Question 6** *Let $u(t,x)$ be given by the following function*

$$u(t,x) = \begin{cases} 1 & \text{if } x \in (0, t/2), \\ -1 & \text{if } x \in (-t/2, 0), \\ 0 & \text{elsewhere.} \end{cases}$$

*Prove that $u$ is a weak solution of Burgers equation $u_t + \partial(u^2/2) = 0$ with zero initial data (i.e. $u(0,x) = 0$).*

**Question 7** *Given an example of a non-identically-zero bounded function $u : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ such that $u(t,x) = 0$ if $t \leq 0$ or $t \geq 1$, and $u$ is a weak solution of the Burgers equation everywhere (but not an entropy solution).*

## 3 Entropy solutions

**Note.** I edited these questions to apply to solutions to scalar conservation law equations in arbitrary dimension.

**Question 8** *Describe explicitly the unique entropy solution to the Burgers equation $u_t + u\,u_x = 0$ with initial data*

$$u(0,x) = \begin{cases} 1 & \text{if } x \in [0,1], \\ 0 & \text{otherwise.} \end{cases}$$

**Hint.** The function $u(t,x)$ is equal to 1 in some region, to $x/t$ in another, and to 0 elsewhere. You need to determine the curves separating these domains using the Rankine-Hugoniot condition.

**Question 9** *Let $u$ and $v$ be two entropy solutions of the conservation law equation*

$$u_t + \operatorname{div} f(u) = v_t + \operatorname{div} f(v) = 0.$$

*Assume that $u, v \in C([0,\infty), L^1_{loc}(\mathbb{R}^d))$, in the sense that the equation (18) in the notes holds. Assume that both bounded functions $u$ and $v$ take values in an interval $I \subset \mathbb{R}$ and let $M = \max\{f'(w) : w \in I\}$. Prove that for any $a < b$, the quantity*

$$D(t) = \int_{B_{R-Mt}} [u(t,x) - v(t,x)]_+ \, \mathrm{d}x,$$

*is monotone decreasing in $t$.*

*Conclude that the initial value problem*

$$u_t + \operatorname{div} f(u) = 0, \qquad u(0,x) = u_0(x)$$

*has the following finite speed of propagation property. The values of $u_0(x)$ for $x \in B_R(x_0)$ determine the values of $u(t,y)$ for all $(t,y)$ such that $|y - x_0| < R - Mt$.*

**Question 10** *Let $u, v \in C([0,\infty), L^1_{loc}(\mathbb{R}^d))$ be two entropy solutions to the same conservation law equation. Prove that if $u(0,x) \leq v(0,x)$ for all $x \in \mathbb{R}$, then $u(t,x) \leq v(t,x)$ for (almost) all $t > 0$ and $x \in \mathbb{R}^d$.*

**Question 11** *Let $u$ and $v$ be two entropy solutions to the same conservation law equation. Prove that $\max(u, v)$ is a weak subsolution of the same equation.*

**Question 12** *Let $u_k$ be a uniformly bounded sequence of entropy solutions of $\partial_t u_k + \operatorname{div} f(u_k) = 0$. Assume that $u_k \to u$ in $L^1_{loc}([0,\infty) \times \mathbb{R}^d)$ in the sense that for any compact subset $K \subset [0,\infty) \times \mathbb{R}^d$ we have*

$$\lim_{k\to\infty} \iint_K |u_k(t,x) - u(t,x)| \; \mathrm{d}x \; \mathrm{d}t = 0.$$

*Prove that $u$ is also an entropy solution of the same equation.*

**Question 13** *Given any convex function $\eta : \mathbb{R} \to \mathbb{R}$, we define the entropy dissipation measure as*

$$\mu = -\eta(u)_t - \operatorname{div} q(u).$$

*By definition, this is a nonnegative measure in $[0,\infty)\times\mathbb{R}^d$. Prove that it is absolutely continuous with respect to $\mathcal{H}^d$ (that is the Haussdorff measure of dimension $d$ for subsets in $\mathbb{R}^{d+1}$).*

**Question 14** *The space of functions of bounded variation is by definition given by the functions for which their BV norm is finite:*

$$\|f\|_{BV} = \|f\|_{L^1} + \sup_{h\in\mathbb{R}^d} \frac{\|f - f(\cdot - h)\|_{L^1}}{|h|}.$$

*(a) Prove that if the initial data $u_0 \in BV$, and $u \in C([0,\infty), L^1(\mathbb{R}^d)) \cap L^\infty([0,\infty) \times \mathbb{R}^d)$ is an entropy solution, then $u(t,\cdot) \in BV$ for all $t > 0$ and $\|u(t,\cdot)\|_{BV} \leq \|u_0\|_{BV}$.*

*(b) Prove that $f \in BV$ if and only $f$ its distributional derivatives are signed measures with finite total variation and*

$$\|f\|_{BV} \approx \|f\|_{L^1} + \sum_i |\partial_i f|.$$

**Note.** Part (b) in Question 14 is just for general knowledge. The first definition given above for the BV norm is all we need for now.

## 4 The vanishing viscosity method

**Question 15** *Let $u_0 \in L^\infty \cap L^1(\mathbb{R}^d)$. Assume $f$ is $C^\infty$. Prove that for every $\varepsilon > 0$, there exists a solution $u$ to the equation*

$$\begin{aligned} \partial_t u + \operatorname{div} f(u) - \varepsilon\Delta u &= 0, \\ u(0,x) &= u_0(x). \end{aligned} \tag{1}$$

*So that $u(t,\cdot) \to u_0$ in $L^1(\mathbb{R}^d)$ as $t \to 0$ and $u$ is $C^\infty$ for all $t > 0$.*

*Note that if $f$ is less regular than $C^\infty$, we also get a solution $u_\varepsilon$ that will be correspondingly less regular.*

**Hint.** After setting up the proof as a fixed point by the contraction mapping theorem, note that the amount of time that we can allow the equation to evolve depends on the $L^1$ norm of the initial data only. A posteriori, justify that the $L^1$ norm of the solution is non-increasing in time, so the intervals of time that we increase in each application of the contraction mapping does not shrink to zero.

**Question 16** *Let $u$ and $v$ be solutions of (1), for the same value of $\varepsilon > 0$, with different initial data. Prove that*

$$\int_{\mathbb{R}^d} |u(t,x) - v(t,x)| \; \mathrm{d}x,$$

*is non-increasing in time.*

*In particular, if for some modulus of continuity $\omega$ the initial data satisfies*

$$\int_{\mathbb{R}^d} |u_0(x) - u_0(x+h)| \; \mathrm{d}x \leq \omega(|h|),$$

*then we also have for all $t > 0$,*

$$\int_{\mathbb{R}^d} |u(t,x) - u(t,x+h)| \, \mathrm{d}x \le \omega(|h|).$$

**Question 17** *The purpose of this question is to obtain a modulus of continuity with respect to time for solutions of* (1).

*(a) Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be a bounded smooth function. Let $u$ be a solution of* (1). *Verify that*

$$\left| \int_{\mathbb{R}^d} \varphi(x)(u(t+h,x) - u(t,x)) \, \mathrm{d}x \right| \le h \left( |\nabla \varphi|_{L^\infty} [f]_{Lip} + \varepsilon \|\Delta \varphi\|_{L^\infty} \right) \|u\|_{L^\infty_t L^1_x}.$$

*This implies that the function $u$ is weakly continuous respect to $t$ independently of $\varepsilon$.*

*(b) Let $\eta_\delta$ be a standard mollifier supported in $B_\delta$. Let us apply part (a) to $\varphi = [u(t,\cdot) - u(t,\cdot+h)] * \eta_\delta$ and obtain*

$$\left| \int_{\mathbb{R}^d} \varphi(x)(u(t+h,x) - u(t,x)) \, \mathrm{d}x \right| \lesssim \left( \frac{h}{\delta} + \frac{\varepsilon h}{\delta^2} \right) \|u\|_{L^1}.$$

*(c) Prove that*

$$0 \le \int_{\mathbb{R}^d} |u(t+h,x) - u(t,x)| - \varphi(x)(u(t+h,x) - u(t,x) \, \mathrm{d}x \le 2\omega(\delta),$$

*where $\omega$ is the modulus of integrability of the initial data like in the previous question.*

*(d) Conclude that there is another modulus of continuity $\tilde{\omega}$, depending only on $\omega$, $[f]_{Lip}$ and $\|u_0\|_{L^1}$, so that for any $\varepsilon \in (0,1)$, $t > 0$ and $h > 0$,*

$$\left| \int_{\mathbb{R}^d} |u(t+h,x) - u(t,x)| \, \mathrm{d}x \right| \le \tilde{\omega}(h).$$

**Question 18** *Using the previous questions, justify that for any initial data $u_0 \in L^\infty \cap L^1(\mathbb{R}^d)$, the solutions to the equation* (1) *for $\varepsilon \in (0,1)$ satisfy the following three properties.*

- *They are uniformly bounded in $L^\infty$ by $\|u_0\|_{L^\infty}$.*
- *They are equi-continuous in $t$ with values in $L^1(\mathbb{R}^d)$.*
- *For all $t > 0$, they take values in a compact subset of $L^1(\mathbb{R}^d)$.*

*Consequently, there exists a subsequential limit $u = \lim u_{\varepsilon_k}$ with $\varepsilon_k \to 0$ that is an entropy solution of the inviscid conservation law equation.*

$$u_t + \operatorname{div} f(u) = 0.$$

**Question 19** *Given $u_0 \in L^\infty(\mathbb{R}^d)$, prove that there exists a unique entropy solution $u \in L^\infty([0,\infty) \times \mathbb{R}^d) \cap C([0,\infty), L^1_{loc}(\mathbb{R}^d))$ of*

$$u_t + \operatorname{div} f(u) = 0,$$
$$u(0,x) = u_0.$$

**Question 20** *Prove that the solutions $u_\varepsilon$ to* (1) *converge to the solution $u$ to the inviscid problem as $\varepsilon \to 0$. In other words, it is not necessary to take a subsequence.*

## 5 Semicontinuous envelopes

**Question 21** *Let $u \in L^\infty_{loc}(\mathbb{R}^d)$. For any $r > 0$, let us define the functions $\overline{u}_r$ and $\underline{u}_r$ by the formula*

$$\overline{u}_r(x) = \operatorname{esssup}_{B_r(x)} u, \qquad \underline{u}_r(x) = \operatorname{essinf}_{B_r(x)} u.$$

(a) *Prove that*

$$\overline{u}_r = \lim_{k\to\infty} \sup\{u(x + ri/k) : i = -k, \dots, -1, 0, 1, \dots, k\} \qquad a.e.,$$
$$\underline{u}_r = \lim_{k\to\infty} \inf\{u(x + ri/k) : i = -k, \dots, -1, 0, 1, \dots, k\} \qquad a.e..$$

(b) *Let $\overline{u} = \lim_{r\to 0} \overline{u}_r$ and $\underline{u} = \lim_{r\to 0} \overline{u}_r$. Prove that these limits exists at every point $x \in \mathbb{R}^d$. Moreover, $\overline{u}$ is upper semicontinuous and $\underline{u}$ is lower semicontinuous.*

(c) *Give an example of a function $u \in L^\infty(\mathbb{R}^d)$ such that $\overline{u}$ is* ***not*** *equal to $u$ almost everywhere.*

(d) *Prove that $\overline{u}$ is the smallest upper semicontinuous function which is larger or equal to $u$ almost everywhere. Correspondingly, $\underline{u}$ is the larger lower semicontinuous function which is smaller or equal to $u$ almost everywhere.*

(d) *Prove that $u$ is almost everywhere equal to a continuous function in a set $D \subset \mathbb{R}^d$ if and only if $\overline{u} = \underline{u}$ in $D$.*

**Question 22** *Let $u : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ be an entropy solution to a scalar conservation law*

$$u_t + \operatorname{div} f(u) = 0.$$

*Let $\overline{u}_r$ and $\underline{u}_r$ be, for each fixed $t$, as in Question 21.*

(a) *Prove that $\overline{u}_r$ is an entropy subsolution and $\underline{u}_r$ is an entropy supersolution.*

(b) *For $\alpha > 0$, let us define the following seminorm, for $f \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$,*

$$[f]_\alpha := \sup_{r>0} \frac{1}{r^\alpha} \|\overline{f}_r - \underline{f}_r\|_{L^1}.$$

*Prove that $[u(t, \cdot)]_\alpha$ is non-increasing in time. In particular, for a smooth enough initial data $u_0$ decaying appropriately at infinity, we will have $[u(t, \cdot)]_1$ uniformly bounded.*

(c) *Prove that if $[f]_\alpha < +\infty$ for some function $f : \mathbb{R}^d \to \mathbb{R}$, then $f$ is continuous except on a set of Haussdorff dimension at most $d - \alpha$.*

**Question 23** *Let $u : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ be an entropy solution to a scalar conservation law*

$$u_t + \operatorname{div} f(u) = 0.$$

*Assume that for some $t_0 \in [0, \infty)$, the function $u(t_0, \cdot)$ is continuous at the point $x_0 \in \mathbb{R}^d$. Prove that the function $u$ is continuous in space-time at the point $(t_0, x_0)$.*

**Question 24** *Let $u : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ be an entropy solution to a scalar conservation law*

$$u_t + \operatorname{div} f(u) = 0.$$

*Prove that if its initial value $u_0$ is continuous almost everywhere, then $u$ is also continuous almost everywhere.*

# 6 Genuinely nonlinear equations

We will deduce some surprising regularization effects for entropy solutions to nonlinear conservation law equations. We first need a notion of a genuinely nonlinear equation.

**Definition 1** *We say an equation $a(u) \cdot \nabla u = 0$ is* genuinely nonlinear *with order $\alpha > 0$ if for every $\xi \in \partial B_1$ we have*

$$|\{v \in \mathcal{I} : |a(v) \cdot \xi| < \delta\}| \lesssim \delta^\alpha.$$

*Here $\mathcal{I}$ denotes an interval of real numbers where the function $u$ takes its values. Since we always work with solutions in $L^\infty$, we can always reduce our analysis to a finite interval of possible values for $u$.*

**Question 25** *Write an example of a genuinely nonlinear function $a : [0,1] \to \mathbb{R}^3$.*

**Question 26** *Assume the equation is genuinely nonlinear. Prove that the values of $a(v)$ for $v$ in any small interval $(r,s) \subset \mathcal{I}$ linearly span the full space $\mathbb{R}^d$.*

**Question 27** *If $a$ is smooth, prove that the genuine nonlinearity condition is equivalent to the fact that there is an integer $k > 0$ such that for all $v \in \mathcal{I}$, the vectors*

$$\{a(v), a'(v), a''(v), \dots, a^{(k)}(v)\}$$

*span the full space $\mathbb{R}^d$.*

**Question 28** *Prove that there is no genuinely nonlinear equation in dimension $d$ with $\alpha > 1/d$.*

# 7 Kinetic formulation

**Question 29** *Given any function $u(t,x)$, we build the following auxiliary function $\chi(t,x,v)$.*

$$\chi(t,x,v) := \begin{cases} 1 & \text{if } 0 \le v < u(t,x), \\ -1 & \text{if } u(t,x) \le v < 0, \\ 0 & \text{otherwise.} \end{cases}$$

*Prove that $u$ is an entropy solution of $u_t + \operatorname{div} f(u) = 0$ for $(t,x) \in \Omega$ if and only if there exists a nonnegative Borel measure $m$ in $\Omega \times \mathbb{R}$, whose support is bounded in $v$, such that*

$$\chi_t + a(v) \cdot \nabla \chi = \partial_v m,$$

*where $a(v) = f'(v)$. Moreover, the following relations hold*

- $u = \int \chi \, \mathrm{d}v$.
- *For any entropy pair $(\eta, q)$, the entropy dissipation measure $\mu = -\eta(u)_t - \operatorname{div} q(u)$ is equal to the integral of $m$ against $\eta''(v)$ with respect to $v$. That is, for any test function $\varphi(t,x)$,*

$$-\iint (\eta(u)_t + \operatorname{div} q(u))\varphi \, \mathrm{d}x \, \mathrm{d}t = \iiint \eta''(v)\varphi(t,x) \, \mathrm{d}m.$$

At this point, the role played by the time variable is irrelevant. It is more convenient to consider the equation is stationary form. There is no loss of generality since we can think of the time variable as another spatial coordinate by writing the equation in one more dimension like

$$(1, a(u)) \cdot (\partial_t, \nabla_x) u = 0.$$

Without a time variable, the kinetic formulation of the equation is slightly shorter to write

$$\chi(x,v) := \begin{cases} 1 & \text{if } 0 \le v < u(x), \\ -1 & \text{if } u(x) \le v < 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

The function $u$ is an entropy solution of $\operatorname{div} f(u) = a(u) \cdot \nabla u = 0$ in some domain $\Omega \subset \mathbb{R}^d$ if and only if

$$a(v) \cdot \nabla \chi(x, v) = \partial_v m \qquad \text{in } \Omega \times \mathbb{R},$$

for some Borel measure $m \geq 0$.

By analogy with kinetic equations in statistical mechanics (i.e. Boltzmann, Vlasov or Landau equations), we call this reformulation of the notion of entropy solutions the *kinetic formulation* of conservation laws.

**Question 30** *Consider the following modified kinetic function* $\tilde{\chi}$

$$\tilde{\chi}(x, v) := \begin{cases} 1 & \text{if } v < u(x), \\ 0 & \text{if } u(x) \leq v. \end{cases}$$

*Prove that* $u$ *is an entropy solution of* $a(u) \cdot \nabla u = 0$ *if and only if* $a(v) \cdot \nabla_x \tilde{\chi} = \partial_v m$ *for some Borel measure* $m \geq 0$ *whose support is bounded in* $v$.

**Question 31** *Let* $u : B_R \to \mathbb{R}$ *be an entropy solution to the conservation law* $a(u) \cdot \nabla u = 0$. *Consider its kinetic formulation as in* (2). *Prove that for* $r < R$,

$$m(B_r \times \mathbb{R}) \lesssim \frac{(max|a|)}{R - r} \int_{B_R} |u|^2 \, dx.$$

**Question 32** *Prove that if* $u \in C^{1/2+\varepsilon}$ *for any* $\varepsilon > 0$, *then the kinetic entropy dissipation measure* $m$ *vanishes.*

**Note.** It is actually the case that if $u$ is continuous, then $m \equiv 0$, but it will take a bit longer to prove.

**Question 33** *Let* $\chi$ *and* $m$ *be as in* (2). *Assume that* $m \equiv 0$. *Thus, the function* $\chi$ *solves*

$$a(v) \cdot \nabla \chi = 0,$$

*in the sense of distributions. Prove that (after modification in a set of measure zero if necessary),* $\chi$ *satisfies the identity*

$$\chi(x) = \chi(x + ta(v)),$$

*pointwise for any points* $x \in \mathbb{R}^d$, $v \in \mathbb{R}$ *and* $t \in \mathbb{R}$ *so that the full segment* $x + sa(v)$ *is inside the domain of the equation for* $s \in [0, t]$.

**Question 34** *Assume* $a$ *is genuinely nonlinear. Let* $u : \mathbb{R}^d \to \mathbb{R}$ *be a global bounded entropy solution of* $a(u) \cdot \nabla u = 0$ *so that* $m = 0$ *in* (2). *Prove that* $u$ *is constant.*

**Hint.** If $u(x) > u(y)$, find points $u(y) < v_1 < \cdots < v_d < u(x)$ so that $a(v_1), \ldots, a(v_d)$ span $\mathbb{R}^d$. Construct a polygonal joining $x$ with $y$ following these directions and track the values of the kinetic function $\chi$ along this polygonal.

**Question 35** *Assume* $a$ *is genuinely nonlinear. Let* $u : B_2 \to \mathbb{R}$ *be a bounded entropy solution of* $a(u) \cdot \nabla u = 0$ *so that* $m = 0$ *in* (2). *Prove that* $u$ *is Hölder continuous in* $B_1$ *(with a Hölder exponent depending on* $\alpha$*).*

**Hint.** If $x, y \in B_1$ are sufficiently close to each other depending on $u(x) - u(y)$, the polygonal constructed in the previous question will stay inside $B_2$. Quantifying this properly may require some nontrivial linear algebra.

**Note.** The previous question shows the regularization effect of the kinetic formulation in its most basic form: when the right hand side is equal to zero. We will see that the equation

$$a(v) \cdot \nabla \chi = \ldots$$

has a subtle regularization effect depending on what is on the right hand side. When we have the derivative of a measure $\partial_v m$ on the right hand side, we will get that $u \in W^{s,1}$ for all $s < \alpha/(2+\alpha)$. This is in some way optimal for the kinetic equation (2), but it is not an optimal result for conservation laws. It is conjectured that $u \in W^{s,1}$ for all $s < \alpha$. This conjectured optimal regularity is known to hold for the Burgers equation.

# 8 Averaging lemmas

In this section of problems we study generic solution of the transport equation

$$a(v) \cdot \nabla_x f = \partial_v g. \tag{3}$$

We could study more generic situation in which the right-hand side is not necessarily the derivative in $v$ of a function/measure $g$. Since we aim at applying the results to conservation laws, let us stick to this one formulation. However, we do not assume that $g$ is a nonnegative measure or that $f$ has any particular form. We will consider $f$ and $g$ to be compactly supported in $x$ and $v$ in order to keep some technicalities cleaner. Our purpose is to study the regularity of $\int f \, \mathrm{d}v$ in terms of the regularity of $f$ and $g$. As we will see, it turns out that we can often prove that $\bar{f}$ is more regular than $f$. This was observed by the mid 80's in the work of François Golse in the context of kinetic equations (as in the equations of statistical mechanics). Because of the kinetic formulation of conservation laws, we will be able to take advantage of these averaging results and get somewhat surprising regularization effects.

Here are some basic facts that will be used below. We first recall that fourier multipliers correspond to convolution operators:

$$\left(\phi(\xi)\hat{f}(\xi)\right)^{\vee} = \check{\phi} * f.$$

It also works in specific directions. For $e \in \partial B_1$ and $\phi : \mathbb{R} \to \mathbb{R}$,

$$\left(\phi(\xi \cdot e)\hat{f}(\xi)\right)^{\vee} = \int_{-\infty}^{\infty} \check{\phi}(ze) * f(x - ze) \, \mathrm{d}z.$$

Young's inequality helps us estimate the $L^p$ norms of convolutions.

$$\|f * g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q} \qquad \text{for } 1 + 1/r = 1/p + 1/q.$$

We start by describing the Littlewood-Paley decomposition of generic functions. We start with some smooth nonnegative bump function $\psi_0 : \mathbb{R}^d \to \mathbb{R}$ supported in $B_1$ such that it equals one in $B_1$. For $j = 1, 2, \dots$, et $\psi_j(\xi) = \psi_0(2^{-j}\xi) - \psi_0(2^{-j+1}\xi)$. For any $f : \mathbb{R}^d \to \mathbb{R}$, e define $\Delta_j f$ to be the function whose Fourier transform equals

$$(\Delta_j f)^{\wedge} = \psi_j(\xi)\hat{f}(\xi).$$

Each term $\Delta_j f$ in the Littlewood-Paley decomposition is localized in a dyadic annulus in frequency space. This decomposition is sometimes handy in order to rephrase some properties of functions. Intuitively, the regularity of $f$ corresponds to the decay in the norm of the terms $\Delta_j f$. Note that each term $\Delta_j f$ can also be expressed as the convolution $\check{\psi}_j * f$ and $\|\check{\psi}_j\|_{L^1}$ is a constant for $j = 1, 2, 3, \dots$.

**Question 36** *Given $f \in L^2(\mathbb{R}^d)$, prove that*

*(a)* $f = \sum_{j=0}^{\infty} (\Delta_j f)$ *(with convergence in $L^2$).*

*(b)* $\|f\|_{L^2}^2 \approx \sum_{j=0}^{\infty} \|\Delta_j f\|_{L^2}^2$.

*(c) If $f$ and $g$ satisfy* (3) *and $f_j = \Delta_j^x f$ and $g_j = \Delta_j^x g$, then*

$$a(v) \cdot \nabla_x f_j = \partial_v g_j.$$

*(d) For any $p \in [1, \infty]$, prove that $\|\Delta_j f\|_{L^p} \leq C \|f\|_{L^p}$ with a constant $C$ independent of $j$.*

**Hint.** For part (d), rewrite the operator $\Delta_j$ as a convolution and use Young's inequality.

We will further decompose the solutions to (3). For an arbitrary sequence $\delta_j > 0$, we write

$$f_j = \sum_{k=0}^{\infty} f_{jk},$$

where

$$\hat{f}_{jk}(\xi, v) = \psi_k\left(\frac{a(v)\cdot\xi}{\delta_j}\right)\psi_j(\xi)\hat{f}(\xi, v).$$

Here $\hat{f}(\xi, v)$ denotes the Fourier transform of $f$ in $x$ only (not in $v$). Note we are abusing notation since the $\psi_k$ in the first factor is over $\mathbb{R}$ whereas the $\phi_j$ in the second factor is over $\mathbb{R}^d$.

The idea of this decomposition is that in the support of each $f_{jk}$ with $k > 1$ we have a lower bound for $|a(v)\cdot\xi|$. The equation will be useful to estimate the norm of each $f_{jk}$ with $k > 1$ in terms of $g_j$. for the reminder term $f_{j0}$, we will use the genuine nonlinearity condition to bound its norm using that for each $\xi$ there cannot be too many values of $v$ so that $f_{j0}(\xi, v) \neq 0$.

**Question 37** *Let $T_k$ be the operator whose Fourier multiplier is $\psi_k\left(\frac{a(v)\cdot\xi}{\delta}\right)$. Prove that $T_k$ is bounded from $L^q$ to $L^q$ uniformly with respect to $k$, $v$ and $\delta$. Moreover, the norm of $T_0$ is $\|\check{\psi}_0\|_{L^1}$ and the norm of $T_k$ is $\|\check{\psi}_1\|_{L^1}$ for any $k \geq 1$.*

**Hint.** Rewrite the Fourier multiplier operator as a convolution in one variable and apply the Young's inequality.

**Question 38** *Assume that $a$ is genuinely nonlinear. For any $j \geq 1$, let $\bar{f}_{j0} = \int f_{j0}\,\mathrm{d}v$. Prove the estimates*

*(a)* $\|\bar{f}_{j0}\|_{L^2} \leq C\min(1, \delta_j^{\alpha/2}2^{-\alpha j/2})\|f_j\|_{L^2}$.

*(b)* $\|\bar{f}_{j0}\|_{L^1} \leq C\|f_j\|_{L^1}$.

*(c)* $\|\bar{f}_{j0}\|_{L^p} \leq C\min(1, \delta_j^{\alpha}2^{-\alpha j})^{1/p'}\|f_j\|_{L^p}$ *for any* $p \in [1, 2]$.

*The constant $C$ depends on the parameters of the genuine nonlinearity condition and the length of the support of $f$ in $v$.*

**Hint.** Use the Cauchy Schwartz inequality in Fourier side.

**Question 39** *Verify the following identity*

$$(\bar{f}_{jk})^\wedge(\xi) = i\int \hat{g}(\xi, v)\left(\frac{a'(v)\cdot\xi}{2^k\delta_j}\right)\tilde{\psi}_1'\left(\frac{a(v)\cdot\xi}{2^k\delta_j}\right)\frac{\psi_j(\xi)}{2^k\delta_j}\,\mathrm{d}v,$$

*where $\tilde{\psi}_1 : \mathbb{R}\to\mathbb{R}$ is the function $\tilde{\psi}_1(z) = \psi_1(z)/z$.*

**Hint.** There should be no difficulty in this question. I had to put the formula somewhere.

**Question 40** *For any $a$ in a bounded set, prove that the operators whose Fourier multipliers are*

$$\left(\frac{a'(v)\cdot\xi}{2^j}\right)\psi_j(\xi) \quad \textit{and} \quad \tilde{\psi}_1\left(\frac{a(v)\cdot\xi}{2^k\delta}\right),$$

*are bounded from $L^p$ to $L^p$, for any $p \in [1, \infty]$, with their norms bounded independently of $k$, $j$ and $\delta$.*

**Question 41** *Prove that for any $q \in [1, \infty]$,*

$$\|\sum_{k=1}^{\infty}\bar{f}_{jk}\|_{L^q} \leq C\frac{2^j}{\delta_j^2}\|g\|_{L^q}.$$

*Here, as before, $\bar{f}_j = \int f_j\,\mathrm{d}v$.*

The following question is a general fact about interpolation of $L^p$ spaces. It is a particular case of the $K$-method of real interpolation.

**Question 42** *Let $\varphi : M \to \mathbb{R}$ be any measurable function from a measure space $M$. Let $1 \leq p < q \leq \infty$. Assume that*

$$\inf_{\varphi_0+\varphi_1=f} (\|\varphi_0\|_{L^p} + t\|\varphi_1\|_{L^q}) \leq C_0 t^\theta,$$

*then*

$$\|\varphi\|_{L^{r,\infty}} := \sup\{\lambda^{-1}|\{x : f(x) > \lambda\}|^{1/r} : \lambda > 0\} \lesssim C_0,$$

*for $r$ given by $1/r = \theta/q + (1-\theta)/p$.*

**Question 43** *Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ and $\varphi_j = \Delta_j \varphi$ be its Littlewood-Paley blocks. Is it true that*

$$\|\varphi_j\|_{L^r} \leq C\|\varphi_j\|_{L^{r,\infty}},$$

*for a constant $C$ independent of $j$?*

**Question 44** *Going back to the functions $f = \sum_j f_j$ as before, prove that for $p \in [1,2]$ and $q \in [1,\infty]$,*

$$\|\bar{f}_j\|_{L^r} \leq 2^{-j\theta}\|f\|_{L^p}^{1-\theta}\|g\|_{L^q}^{\theta},$$

*where*

$$\theta = \frac{\alpha}{2p' + \alpha} \quad \text{and} \quad \frac{1}{r} = \theta \cdot \frac{1}{q} + (1-\theta) \cdot \frac{1}{p}.$$

**Hint.** Combine the interpolation result of Question 42 with the estimates from Questions 38 and 41. Pick the optimal choice of $\delta_j > 0$ in terms of all the other parameters.

**Question 45** *Given any function $\varphi : \mathbb{R}^d \to \mathbb{R}$. Assume that*

$$\|\Delta_j \varphi\|_{L^{p,\infty}} \leq C 2^{-j\theta}.$$

*Prove that $\varphi \in W^{s,r}(B_1)$ whenever $s \in (0,\theta]$ and $p \in (1,p]$ and either $s < \theta$ or $r < p$. Here, we use the definition*

$$\|\varphi\|_{W^{s,r}(B_1)} := \left(\iint_{B_1\times B_1} \frac{|\varphi(x)-\varphi(y)|^r}{|x-y|^{d+sr}} \,\mathrm{d}y \,\mathrm{d}x\right)^{1/r} + \|\varphi\|_{L^r(B_1)}.$$

*Verify that the inclusion also holds for all variants of the space of functions with $s$ derivatives in $L^r$ that Isaac can come up with.*

**Question 46** *Let $u : B_2 \to \mathbb{R}$ be an entropy solution of a genuinely nonlinear scalar conservation law*

$$a(u) \cdot \nabla u = 0 \quad \text{in } B_2.$$

*Prove that $u \in W^{s,r}(B_1)$ for all $s < \theta$ and $1/r = (1+\theta)/2$, where $\theta = \alpha/(4+\alpha)$. Moreover, an inequality holds*

$$\|u\|_{W^{s,r}(B_1)} \leq C\|u\|_{L^1(B_2)}^{(1+\theta)/2},$$

*for a constant $C$ that depends on dimension, the parameters of the genuine nonlinearity condition, and $\|u\|_{L^\infty}$.*

**Note.** The estimate above is a consequence of averaging results for general kinetic equations. There is a lot of structure which is specific of conservation law equations that we are not using. In particular, we do not use that $f$ only takes the values $+1$, $-1$ and $0$ and has the specific form of Question 29. We do not use that $m \geq 0$ either. The best regularity result that is currently known is that $f \in W^{s,1}$ for all $s \in (0, \alpha/(2+\alpha))$. This result is proved using averaging lemmas as above, combined with some of the extra information about conservation laws. It is conjectured that the actual threshold is $s \in (0,\alpha)$.

## 9 De Giorgi iteration

**Question 47** *Prove that $u : \Omega \to \mathbb{R}$ is an entropy subsolution of the equation $a(u) \cdot \nabla u \leq 0$ if and only if there exist two nonnegative measures $m_0$ and $m_1$ in $\Omega \times \mathbb{R}$ such that the function $\chi$ defined in Question 29 satisfies*

$$a(v) \cdot \nabla_x \chi = \partial m_0 - m_1.$$

**Question 48** *Let $u : B_R \to [0, +\infty)$ be a non-negative entropy subsolution of the equation $a(u) \cdot \nabla u \leq 0$. Prove that there are measures $m_0$ and $m_1$ as in Question 47 that satisfy the bounds*

*(a)*

$$m_0(B_r \times \mathbb{R}) \leq \frac{C}{R-r} \|u\|_{L^1(B_R)} \|u\|_{L^\infty(B_R)} \qquad \text{and} \qquad m_1(B_r \times \mathbb{R}) \leq \frac{C}{R-r} \|u\|_{L^1(B_R)}.$$

*Here $C$ is a constant depending on $a$ only.*

*(b)*

$$m_0(B_r \times \mathbb{R}) \leq C \limsup_{h \to 0} \frac{\|u(\cdot - h) - u\|^2_{L^2(B_{R-|h|})}}{|h|} \qquad \text{and} \qquad m_1(B_r \times \mathbb{R}) \leq \frac{C}{R-r} \|u\|_{L^1(B_R)}.$$

*Here $C$ is a constant depending on $a$ only.*

**Note.** The measures $m_0$ and $m_1$ of Question 47 are not unique. That is why in Question 48 you are supposed to show that the bounds holds for some pair and not all.

**Question 49** *Let $u : B_R \to [0, +\infty)$ be a non-negative entropy subsolution of the equation $a(u) \cdot \nabla u \leq 0$ with $a$ genuinely nonlinear. Use averaging to justify the following inequalities*

*(a) For any $\varphi : \mathbb{R}^d \to \mathbb{R}$ supported inside $B_R$,*

$$\|u\varphi\|_{W^{s,r}} \leq C \|u\|^{\frac{1+\theta}{2}}_{L^1(B_R)} \|\nabla \varphi\|^\theta_{L^\infty},$$

*for any $s < \theta$ and for some constant $C$ depending on $\|u\|_{L^\infty}$ and $a$. The constants $\theta$ and $r$ should be the same as in Question 46.*

*(b) For $p > 1$ such that $1/p > (1+\theta)/2 - \theta/d$,*

$$\|u\|_{L^p(B_r)} \leq C \|u\|^{\frac{1+\theta}{2}}_{L^1(B_R)} (R-r)^{-\theta}.$$

***Hint.*** *You need to use the embedding $W^{s,r} \subset L^p$ for fractional Sobolev spaces, which you may not have seen before.*

*(c) For any $p' > 1$ such that $1/p' < (1-\theta)/2 + \theta/d$, we have*

$$\|u\|_{L^1(B_r)} \leq C (R-r)^{-\theta} \|u\|^{\frac{1+\theta}{2}}_{L^1(B_R)} |\{u > 0\} \cap B_r\}|^{1/p'}.$$

*(d) Verify that the values of $\frac{1+\theta}{2} + 1/p' > 1$ for some valid values of $p'$ in the previous inequality.*

**Note.** It is interesting to take a moment and think of the significance of the result in the last question. For any exponents $\alpha$ and $\beta$ such that $\alpha + \beta \leq 1$, the inequality

$$\|u\|_{L^1(\Omega)} \leq C \|u\|^\alpha_{L^1(\Omega)} |\{u > 0\} \cap \Omega|^\beta$$

is satisfied for some constant $C$ depending on $\|u\|_{L^\infty}$ and $\Omega$ by any function $u$ (regardless of the equation). The same inequality is not necessarily true, and thus it must be obtained as a consequence of the equation, as soon as $\alpha + \beta > 1$.

**Question 50** *Let $u : B_2 \to [0, +\infty)$ be a nonnegative subsolution to a genuinely nonlinear conservation law. The purpose of this question is to prove the following estimate*

$$\operatorname{esssup}_{B_1} u \leq C\|u\|_{L^1(B_2)}^{\gamma}, \tag{4}$$

*for some $\gamma > 0$ and a constant $C$ depending on the parameters of the genuine nonlinearity condition, $\|a\|_{C^1}$, and $\|u\|_{L^\infty(B_2)}$.*

(a) *For an arbitrary constant $U > 0$, let us define*

$$\begin{aligned} r_k &= 1 + 2^{-k}, \\ \ell_k &= U(1 - 2^{-k}), \\ u_k &= (u - \ell_k)_+, \\ a_k &= \|u_k\|_{L^1(B_{r_k})}. \end{aligned}$$

*Show that $\operatorname{esssup}_{B_1} u \leq U$ if and only if $a_k \to 0$ as $k \to \infty$.*

(b) *Prove the following recurrence relation*

$$a_{k+1} \leq C2^{Ck} a_k^{(1+\theta)/2+1/p'} U^{-1/p'}.$$

(c) *Let $\delta := (1 + \theta)/2 + 1/p' - 1$ and $\gamma = \delta p'$. Prove that if $U \geq C a_0^{\gamma}$ for some large $C$, then $a_k \to 0$ as $k \to \infty$.*

(d) *Conclude the proof of (4).*

**Question 51** *Let $u_j$ be a uniformly bounded sequence of entropy solutions to a genuinely nonlinear conservation law that converges in $L^1_{loc}$ to a constant. Prove that it also converges to a constant locally uniformly.*

## 10 Jump set

**Question 52** *Let $m$ be the entropy dissipation measure as in Question 29. Let us define the jump set $J$ as*

$$J := \left\{ x : \limsup_{r \to 0} \frac{m(B_r(x) \times \mathbb{R})}{r^{d-1}} > 0 \right\}.$$

*Prove that the Haussdorff dimension of $J$ is at most $d - 1$.*

**Question 53** *Let $u : \Omega \to \mathbb{R}$ solve a genuinely nonlinear conservation law equation. For any fixed point $x \in \Omega$, let us define a sequence of rescalings*

$$u_r(y) = u(x + ry).$$

*Prove that $u_r$ solves the same conservation law equation independently of the value of $r$. Moreover, there is a subsequence $r_k \to 0$ so that $u_{r_k}$ converges in $L^1_{loc}$ to what we call a* blow-up limit at *$x$. Consequently, blow-up limits also solve the same equation.*

**Question 54** *Prove that any blow-up limit of a solution to a genuinely nonlinear conservation law equation at a point outside of the jump set $J$ must be constant.*

**Question 55** *We say that a function $u \in L^1_{loc}(\Omega)$ is VMO (Vanishing Mean Oscillation) at the point $x$ if*

$$\lim_{r \to 0} \frac{1}{|B_r|} \int_{B_r(x)} |u(y) - m(r)| \, \mathrm{d}y = 0,$$

*where $m : (0, r_0) \to \mathbb{R}$ is the average $m(r) = |B_r|^{-1} \int_{B_r(x)} u(y) \, \mathrm{d}y$.*

*Prove that any entropy solution $u$ to a genuinely nonlinear conservation law is VMO at any point outside of the jump set $J$.*

**Question 56** *Give an example of a bounded function $u : B_1 \to \mathbb{R}$ so that $u$ is VMO at every point, but not all points are Lebesgue points. Give an example of a bounded function $u : B_1 \to \mathbb{R}$ so that every point is a Lebesgue point but the function is not continuous.*

**Question 57** *Prove that any entropy solution $u$ to a genuinely nonlinear conservation law is continuous at any point outside of $J$.*

**Question 58** *Prove that for any solution of a genuinely nonlinear conservation law we have*

$$J := \left\{ x : \liminf_{r \to 0} \frac{m(B_r(x) \times \mathbb{R})}{r^{d-1}} > 0 \right\}.$$