Boot camp - Problem set

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In the summer of 2017, I led an intensive study group with four undergraduate students at the University of Chicago (Matthew Correia, David Lind, Jared Marx-Kuo, and Isaac Neal). They read Luis Caffarelli's notes *The obstacle problem* from the Fermi lectures 1998. This problem set was given to complement those readings and introduce more material.

1 Week 1

1.1 Variational analysis

The purpose of the following questions is to obtain a standard theorem from variational analysis. Most of these steps are done somewhere in Brezis' book.

Question 1. *Prove that a closed convex set in a Banach space is weakly closed.*

Definition 1.1. *A function $f : X \to \mathbb{R}$, where X is any metric (or topological) space is said to be lower semicontinuous when $f^{-1}(\{x : x > a\})$ is open for every $a \in \mathbb{R}$.*

Equivalently, $f^{-1}(\{x : x \leq a\})$ is closed for every $a \in \mathbb{R}$.

Equivalently (at least for Hausdorff spaces), $f(x) \leq \lim_{y \to x} f(y)$ for all $x \in X$.

Question 2. *Prove that any lower semicontinuous function on a compact set attains its minimum.*

Question 3. *Let X be a Banach space and $f : X \to \mathbb{R}$ be continuous (with respect to the norm topology) and convex. Prove that f is lower semicontinuous with respect to the weak topology.*

Definition 1.2. *We say $f : X \to \mathbb{R}$ is coercive if*

$$\lim_{R \to \infty} \inf\{f(x) : \|x\| \geq R\} = +\infty.$$

Question 4. *Let X be a Banach space and $f : X^* \to \mathbb{R}$ be coercive and weak-$*$ lower semicontinuous. Prove it attains its global minimum.*

Question 5. *Prove that any convex coercive continuous function on a closed convex subset of a reflexive Banach space attains its minimum.*

The result in the last question is restricted to reflexive spaces. This is to put together the facts that closed convex sets are closed in the weak topology, and closed and bounded sets are compact in the weak-$*$ topology. It is not an artifact of the proof, as the following problem shows.

Question 6. *Give an example of a convex, continuous and coercive function $f : L^1(\mathbb{R}) \to \mathbb{R}$ which does not attain its minimum.*

1.2 Variational problems for elliptic PDE

Question 7. *Given $\Omega \subset \mathbb{R}^d$ bounded and Lipschitz, $g \in L^2(\Omega)$ and $f \in H^1(\Omega)$, define the function*

$$J(u) = \int_\Omega a_{ij}(x) \partial_i u \partial_j u + g(x)\, u(x) \, \mathrm{d}x.$$

We assume that the coefficients a_{ij} are uniformly elliptic. That means that for each index i, j, we have $a_{ij} \in L^\infty$ and moreover, there is a constant $\lambda > 0$ so that

$$\{a_{ij}(x)\} \geq \lambda \mathrm{I} \qquad a.e.$$

Verify J attains its minimum in the set

$$\{u \in H^1(\Omega) : u = f \in \partial\Omega\}.$$

Moreover, the minimizer satisfies the following equation in the sense of distributions

$$\partial_i \left(a_{ij}(x)\partial_j u\right) = \frac{1}{2} g \qquad in\ \Omega.$$

Here, repeated indexes denote summation.

Question 8. *Given $f \in H^1(\Omega)$ and $\varphi : \Omega \to \mathbb{R}$ measurable, prove that the set*

$$\{u \in H^1(\Omega) : u - f \in H_0^1(\Omega) \text{ and } u \geq \varphi \text{ a.e. in } \Omega\},$$

is closed and convex in Ω.

Question 9. *Given $\Omega \subset \mathbb{R}^d$ bounded and Lipschitz, and $f \in H^1(\Omega)$. Prove that the following two minimization problems*

$$\min \left\{ \int_\Omega |\nabla u|^2 + u_+ \, \mathrm{d}x : u - f \in H_0^1(\Omega) \right\},$$
$$\min \left\{ \int_\Omega |\nabla u|^2 + u \, \mathrm{d}x : u - f \in H_0^1(\Omega) \text{ and } u \geq 0 \right\},$$

are achieved at the same function.

Here $u_+ = u$ if $u > 0$, and $u_+ = 0$ otherwise. Compute the Euler-Lagrange equation for the minimizer.

Question 10. *Prove that if $u, v \in H^1(\Omega)$, then $\min(u, v) \in H^1(\Omega)$.*

Question 11. *(a) Prove that if $u, v \in L^1(\Omega)$ are superharmonic, then $\min(u, v)$ is also superharmonic.*

(b) Prove that if $u, v \in H^1(\Omega)$ satisfy $\partial_i a_{ij}(x) \partial_j u \leq 0$ and $\partial_i a_{ij}(x)\partial_j v \leq 0$ in the sense of distributions, then for $w = \min u, v$, we also have

$$\partial_i a_{ij}(x) \partial_j w \leq 0.$$

Question 12. ($*$) *Is there a superharmonic function $u \in L^\infty(B_1)$ which is lower semicontinuous but not continuous?*

1.3 Lipschitz and Hölder spaces

Question 13. *Let $f : \mathbb{R}^{d-1} \to \mathbb{R}$ be a Lipschitz function. Let $S = \{x \in \mathbb{R}^d : x_d < f(x_1, \dots, x_{d-1})\}$. Prove that there is a bi-Lipschitz function $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ so that $\varphi^{-1}(S) = \{x \in \mathbb{R}^d : x_d < 0\}$.*

Question 14. *Prove that $C^\infty(\mathbb{R})$ is not dense in $C^{1,1}(\mathbb{R})$. What is the closure of $C^\infty(\mathbb{R})$ with respect to the norm in $C^{1,1}(\mathbb{R})$?*

2 Week 2

2.1 Hölder spaces

A function $f : \Omega \to \mathbb{R}$ is in the Hölder space $C^\alpha(\Omega)$ if it is continuous and the following norm is finite.

$$\|f\|_{C^\alpha(\Omega)} := \|f\|_{L^\infty(\Omega)} + \sup_{x,y \in \Omega} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$

Here $\alpha \in (0, 1)$. The case $\alpha = 1$ corresponds to Lipschitz functions. The space of Lipschitz functions is sometimes written as $C^{0,1}$.

Question 15. *Let $f \in L^\infty(\Omega)$. Assume that for every ball $B_r(x) \subset \mathbb{R}^d$ which intestects Ω in a set of positive measure, we have*

$$\operatorname{esssup}_{\Omega\cap B_r(x)} f - \operatorname{essinf}_{\Omega \cap B_r(x)} f \leq C r^\alpha.$$

Prove that f is a.e. equal to a Hölder continuous function.

Question 16. *Let $f : B_2 \to \mathbb{R}$ be a bounded function. Assume that there is a $\delta > 0$ so that for every ball $B_r(x) \subset B_2$, the following inequality holds*

$$\operatorname*{osc}_{B_{r/2}(x)} f \leq (1-\delta) \operatorname*{osc}_{B_r(x)} f.$$

Prove that f is Hölder continuous in B_1 for some $\alpha > 0$ depending on δ.

Question 17. *Let $f \in C^\alpha(B_1)$. Prove that*

$$\|f\|_{C^\alpha(B_1)} \leq C \left(\|f\|_{L^1(B_1)} + \sup_{x,y\in\Omega} \frac{|f(x)-f(y)|}{|x-y|^\alpha} \right),$$

for some constant C depending on α only.

Question 18. *Let $f \in C^1(\Omega)$. Prove that the following two definitions of the $C^{1,\alpha}$ norms are equivalent.*

$$\|f\|_{C^{1,\alpha}} = \|f\|_{L^\infty} + \sup_{x,y\in\Omega} \frac{|\nabla f(x) - \nabla f(y)|}{|x-y|^\alpha} \approx \|f\|_{L^\infty} + \sup_{x,y\in\Omega} \frac{|f(x)-f(y)-(x-y)\cdot\nabla f(y)|}{|x-y|^{1+\alpha}}.$$

We say that $f \in C^{1,\alpha}(\Omega)$ when these norms are finite.

When Ω is a ball, prove also that

$$\|f\|_{C^{1,1}} \approx \|f\|_{L^\infty} + \|D^2 f\|_{L^\infty}.$$

Question 19. *Let Ω be a convex domain and $f \in C^{1,\alpha}(\Omega)$. Prove that*

$$\|\nabla f\|_{L^\infty(\Omega)} \leq C \|f\|_{C^{1,\alpha}(\Omega)},$$

where the constant C depends on Ω only.

Note. The interpolation inequality above holds for nonconvex Ω, provided that its boundary is not too wild.

Question 20. *Let $f \in C(\Omega)$. Assume that for any ball $B_r(x)$ contained in Ω, there exists a linear function $\ell(x) = a \cdot x + b$ such that*

$$\sup_{x\in\Omega} |f(x) - \ell(x)| \leq C r^{1+\alpha}.$$

Prove that $f \in C^{1,\alpha}(\Omega)$.

2.2 The Laplace equation

There are some standard properties of the Laplace equation that are typically covered in any PDE class and you should know. These are

- The mean value property, both in spheres and in balls.
- The strong maximum principle.
- The Harnack inequality.
- Existence and uniqueness of solutions of the Dirichlet problem in a bounded domain with nice boundary.
- Liouville's theorem.

- Solvability of the equation $\Delta u = f$ in the full space using the fundamental solution.

Question 21. *Let $u : \Omega \to \mathbb{R}$. Assume that u attains its maximum at a point $x_0 \in \Omega$. Prove that x_0 belongs to the support of Δu.*

Question 22. *Let $\Phi : \mathbb{R}^d \to \mathbb{R}$ be the fundamental solution. Prove that for any $x \in B_1$,*

$$\int_{\partial B_1} \Phi(x-y) \, \mathrm{d}y = \textit{Constant independent of } x.$$

Question 23. *Let $x \in \mathbb{R}^d$. Compute the following integral (explicitly as a function of x),*

$$2d \int_{y \in \mathbb{R}^d} \Phi(y) + x \cdot \nabla \Phi(y) - \Phi(y+x) \, \mathrm{d}y.$$

(it takes me 5 keystrokes to write the correct answer in LaTeX).

Question 24. *Let $u : B_r \to \mathbb{R}$ satisfy the equation*

$$\begin{aligned} u &\leq 0 && \text{on } \partial B_r, \\ \Delta u &\geq -C_0 && \text{in } B_r. \end{aligned}$$

Prove that

$$u \leq \frac{C_0}{2d} r^2 \qquad \text{in } B_r.$$

Question 25. *Prove the following generalization of the Harnack inequality. Let $u : B_{4r} \to \mathbb{R}$ be a nonnegative function that satisfies*

$$\Delta u = f \qquad \text{in } B_{4r}.$$

Then

$$\max_{B_r} u \leq C \left(\min_{B_r} u + \|f\|_{L^\infty} r^2 \right),$$

for some constant C depending on dimension only.

Note. The Harnack inequality above is still true if we replace B_{4r} by B_{2r} or $B_{1.0001r}$. The proof with B_{4r} is marginally easier.

Question 26. *(a) Let $u : B_1 \to \mathbb{R}$ be a convex function. Prove that if $\Delta u \leq C$, then $u \in C^{1,1}(B_1)$.*

(b) Let $u : B_1 \to \mathbb{R}$ be a superharmonic fuction. Assume that there is a constant $N > 0$, so that for every $x \in B_1$, there exists a $b \in \mathbb{R}^d$ such that

$$u(y) \geq u(x) + b \cdot (y-x) - N|y-x|^2 \qquad \text{for all } y \in B_1.$$

Prove that $u \in C^{1,1}(B_1)$.

2.3 The obstacle problem

Given $\varphi : \overline{\Omega} \to \mathbb{R}$ and $f : \partial\Omega \to \mathbb{R}$ so that $f \geq \varphi$ on $\partial\Omega$, the solution of the obstacle problem is the unique function which satisfies the following properties.

$$\begin{aligned} u &= f && \text{on } \partial\Omega, \\ u &\geq \varphi && \text{in } \Omega, \\ \Delta u &\leq 0 && \text{in } \Omega \ \ \text{(i.e. } u \text{ is superharmonic)}, \\ \Delta u &= 0 && \text{in } \{u > \varphi\}. \end{aligned}$$

If Ω has a Lipschitz boundary, f is the boundary value of a function in $H^1(\Omega)$ and φ is any function in $H^1(\Omega)$ (or even rougher perhaps), then the solution to the obstacle problem corresponds to the function where the following minimization is achieved.

$$\min \left\{ \int_\Omega |\nabla u|^2 \, \mathrm{d}x : u \geq \varphi \text{ in } \Omega, \ u - f \in H^1_0(\Omega) \right\}.$$

Question 27. *Prove that the function u where the above minimization is achieved, is indeed the solution to the obstacle problem described previously.*

Question 28. *Prove that if v is any superharmonic function such that $v \geq f$ on $\partial\Omega$ and $v \geq \varphi$, then $v \geq u$.*

Question 29. *Let u be a solution to the obstacle problem in B_1. Assume that $\varphi \in C^{1,1}$ and for some reason we know that $u \in C^{1,1}$ in a neighborhood of the boundary ∂B_1 (this would be the case for example if f is $C^{1,1}$ and $f > \varphi$ on ∂B_1). The following is an alternative strategy to prove that $u \in C^{1,1}$.*

(a) For any $h \in \mathbb{R}^d$ sufficiently small, we would have that the function

$$v_h(x) = \frac{1}{2}\left(u(x+h) + u(x-h)\right) + N|h|^2,$$

is larger than φ and larger than u on $\partial B_{1-\delta}$ for some $\delta > 0$ small. Here N is the maximum between $\|\varphi\|_{C^{1,1}(B_1)}$ and $\|u\|_{C^{1,1}(B_1 \setminus B_{1-2\delta})}$

(b) The function v_h is also superharmonic, and therefore $v_h \geq u$ in $B_{1-\delta}$.

(c) For any $h \in \mathbb{R}^d$ small and $x \in B_{1-\delta}$,

$$\frac{u(x+h) + u(x-h) - 2u(x)}{|h|^2} \geq -2N.$$

Consequently $\partial_{ee} u \geq -2N$ in $B_{1-\delta}$ for any unit vector e.

(d) Using that $\Delta u \leq 0$, we conclude $u \in W^{2,\infty}(B_{1-\delta}) = C^{1,1}(B_{1-\delta})$.

3 Week 3

3.1 Haudorff measure

The usual definition of the m-dimensional Hausdorff measure is the following.

$$H^m(A) = \sup_{\delta>0}\left(\inf\left\{\sum_{i=0}^{\infty}(\operatorname{diam}(U_i))^m : \bigcup_{i=1}^{\infty} U_i \supset A,\ \operatorname{diam}(U_i) < \delta\right\}\right).$$

This defines an outer measure. Using Caratheodory's trick, one defines measurable sets, which should include all Borel sets.

The Hausdorff dimension of a set A is defined as the infimum of those values of m such that $H^m(A) = 0$.

A related concept is the Minkowski content and dimension.

We define the m-dimensional Minkowski content by the formula

$$M^m(A) = \limsup_{r>0} \frac{|A + B_r|}{r^m}.$$

(alternatively, with $\liminf$, but let's not care about that one)

Question 30. *Prove that for any set $A \subset \mathbb{R}^d$ and $0 \leq m \leq d$,*

$$H^m(A) \leq C M^m(A),$$

for some constant C depending on d and m only.

Verify that the opposite inequality does not hold if $m > 0$ (use $A = \mathbb{Q}^d$).

The main result in section 3.1 in the notes is that the free boundary $\partial\{u = \varphi\}$ has finite $(d-1)$-dimensional Hausdorff measure. This is a stronger statement than saying that $\{u = \varphi\}$ is a set of finite perimeter. We define the perimeter of a set by the formula

$$\operatorname{Per}(A) = \sup\left\{\left|\int_A \operatorname{div}\varphi\right| : \varphi \in C^1_c(\mathbb{R}^n;\mathbb{R}^n), ||\varphi||_{L^\infty} \leq 1\right\} \tag{3.1}$$

This is the same as the BV (semi)norm of the function χ_A. Here, BV is the space of functions whose derivatives are finite signed Borel measures.

A set A has finite perimeter when $\text{Per}(A) < +\infty$.

The study of sets of finite perimeter is delicate and has several interesting results. The couple of questions below shows some of the most basic properties.

Question 31. *Prove that if $A \subset \mathbb{R}^d$ is a compact set with C^1 boundary, then*

$$\text{Per}(A) = H^{d-1}(\partial A) = \int_{\partial A} 1 \, \mathrm{d}S.$$

Question 32. *Prove that for any Lebesgue measurable set,*

$$\text{Per}(A) \leq C H^{d-1}(A).$$

In particular, if $H^{d-1}(A) < +\infty$, then A is a set with finite perimeter.

Question 33. *Prove that the set*

$$A = \bigcup_{i=0}^{\infty} B_{2^{-i}}(q_i),$$

where q_i is an enumeration of $\mathbb{Q}^d$, is of finite perimeter. However, the Hausdorff dimension of ∂A is equal to d.

Some finer properties of a set of finite perimeter is that there is a special set called *reduced boundary*, $\partial^* A \subset \partial A$, and a vector field $\nu : \partial^* A \to \partial B_1$, such that

$$\int_A \nabla \varphi \, \mathrm{d}x = \int_{\partial^* A} \varphi(x) \nu(x) \, \mathrm{d}H^{d-1}.$$

Moreover, for every $x \in \partial^* A$, we have that the *blow-up* sequence

$$f_R(y) = \chi_{R(A-x)},$$

converges to a half space (perpendicular to $\nu(x)$) in L^1_{loc} as $R \to \infty$.

Proving this takes more time. It was covered in our recent summer school by Maggi's minicourse. A good source is Maggi's book "Sets of Finite Perimeter and Geometric Variational Problems".

It's easy to verify that for the set A in the last question, the reduced boundary is the union of the spheres $\bigcup_{i=0}^{\infty} \partial B_{2^{-i}}(q_i)$.

Question 34. *Let $A \subset \mathbb{R}^d$ be a Lebesgue measurable set. Assume that there is a constant C such that for all $r \in (0, 1)$,*

$$|(A + B_r) \setminus A| \leq C r^{d-1}.$$

Prove that A has finite perimeter. Is it true that $H^{d-1}(\partial A) \lesssim C$?

Question 35. *Besides the obstacle problem, another classical free boundary problem consists in the following equations*

$$\begin{aligned} u &\geq 0 && \textit{everywhere}, \\ \Delta u &= 0 && \textit{where } u > 0, \\ u_\nu &= 1 && \textit{in } \partial\{u > 0\}. \end{aligned}$$

This problem is obtained by minimizing the (noncovex) functional

$$J(u) = \int |\nabla u|^2 \, \mathrm{d}x + |\{u > 0\}|,$$

for some given nonnegative boundary condition.

Assuming that u is a solution to this problem in B_1 and $u \in C^1(B_1 \cap \overline{\{u > 0\}})$ (obviously, it isn't differeniable on $\partial\{u > 0\}$ if we look outwards, you know what I mean), prove that

$$H^{d-1}(\partial\{u > 0\} \cap B_{1/2}) \leq C,$$

for some constant C.

Hint. Here $|\nabla u| \geq 1 - \varepsilon$ in a neighborhood of $\partial\{u > 0\}$. Moreover, from the assumption we should also be able to deduce that $u(x) \approx \mathrm{dist}(x, \partial\{u > 0\})$. Taking this into account, we have

$$|(\{u = 0\} + B_\delta) \setminus \{u = 0\}| \approx \int_{0<u<\delta} |\nabla u|^2 \, \mathrm{d}x.$$

4 Week 5

4.1 De Giorgi - Nash theory

Throughout this section, we study elliptic equations with variable coefficients $a_{ij}(x)$.

$$\partial_i[a_{ij}(x)\partial_j u] = 0.$$

We use the convention that repeated indexes denote summation. We say that he coefficients a_{ij} are uniformly elliptic if there are two constants $\Lambda \geq \lambda > 0$ so that for every x in the domain of the equation,

$$\lambda \mathrm{I} \leq \{a_{ij}(x)\} \leq \Lambda \mathrm{I}.$$

These are matrix inequalities. For every x, $a_{ij}(x)$ should be a symmetric matrix whose eigenvalues lie between λ and Λ.

We do not make any regularity assumption on the cofficients $a_{ij}(x)$ other than being measurable functions.

Question 36. *For any bounded open set $\Omega \subset \mathbb{R}^d$, prove that*

$$\int_\Omega a_{ij}(x)\partial_i u \partial_j u \, \mathrm{d}x,$$

is equivalent to the usual norm in H_0^1.

Question 37. *Prove that the minimizer of*

$$\min \left\{ \int_\Omega a_{ij}(x)\partial_i u \partial_j u \, \mathrm{d}x : u \in H^1(\Omega), u = f \text{ on } \partial\Omega \right\},$$

is attained by a function that solves the equation

$$\begin{aligned} u &= f \text{ on } \partial\Omega, \\ \partial_i[a_{ij}(x)\partial_j u] &= 0 \text{ in } \Omega. \end{aligned}$$

We say that a function $u : \Omega \to \mathbb{R}$ is a subsolution of the equation $\partial_i[a_{ij}\partial_j] \geq 0$, if for any C^1 function $\varphi : \Omega \to \mathbb{R}$, with $\varphi \geq 0$ and $\varphi = 0$ on $\partial\Omega$, we have

$$\int_\Omega a_{ij}(x)\partial_i u(x)\partial_j \varphi_j(x) \, \mathrm{d}x \leq 0. \tag{4.1}$$

Question 38. *Prove that if $u \in H^1(\Omega)$ is a subsolution, then (4.1) holds for any $\varphi \in H_0^1$.*

Question 39.

- *Let $u \in H^1(\Omega) \cap L^\infty(\Omega)$ and $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Prove that $f \circ u \in H^1(\Omega)$ and $\nabla(f \circ u) = f'(u) \nabla u$.*
- *Let $u \in H^1(\Omega)$. Prove that $u_+ \in H^1(\Omega)$ and*

$$\nabla u^+(x) = \begin{cases} \nabla u(x) & \text{if } u(x) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

- *Prove that $\nabla u = 0$ almost everywhere in the set $\{u = 0\}$.*

- *Prove that if there is a measurable set $A \subset \Omega$ and the function*

$$u(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise,} \end{cases}$$

belongs to $H^1(\Omega)$, then either $A = \emptyset$ or $A = \Omega$.

Question 40.

1. *Prove that if $u \in H^1(\Omega)$ is a subsolution and $F : \mathbb{R} \to \mathbb{R}$ is monotone increasing and convex, then $F \circ u$ is also a subsolution.*

2. *Prove that if $u, v \in H^1(\Omega)$ are both subsolutions, then $\max(u, v)$ is also a subsolution.*

Question 41. *(Cacciopoli's inequality) Let $u \geq 0$ be a subsolution in $B_{1+\delta}$ and $\varphi : B_{1+\delta} \to \mathbb{R}$ be a nonnegative function such that $\varphi = 0$ on $\partial B_{1+\delta}$. Prove that there is a constant $C > 0$, such that*

$$\int_{B_{1+\delta}} \varphi^2 |\nabla u|^2 \, \mathrm{d}x \leq C \int_{B_{1+\delta}} u^2 |\nabla \varphi|^2 \, \mathrm{d}x.$$

In particular $\|\nabla u\|_{L^2(B_1)} \leq C\delta^{-1} \|u\|_{L^2(B_{1+\delta})}$.

We want to prove the following result.

Theorem 4.1. *Let $u : B_2 \to \mathbb{R}$ be a nonnegative subsolution ($\partial_i[a_{ij}(x)\partial_j u] \geq 0$) with a_{ij} uniformly elliptic. Then*

$$\text{ess-sup}_{B_1} u \leq C \|u\|_{L^2(B_2)},$$

for some constant C which depends only of dimension and the ellipticity constants λ, Λ.

Question 42. *Prove that Theorem 4.1 would follow from the following statement: there exists a constant δ_0, which depends only of dimension and the ellipticity constants λ, Λ, so that if $\|u\|_{L^2(B_2)} \leq \delta_0$, then $\|u\|_{L^\infty(B_1)} \leq 1$.*

In order to prove Theorem 4.1, we consider the following setup. Let

$$\begin{aligned} \ell_k &:= 1 - 2^{-k}, \\ r_k &:= 1 + 2^{-k}, \\ u_k &:= (u - \ell_k)_+, \\ A_k &:= \|u_k\|_{L^2(B_{r_k})} \end{aligned}$$

Observe that each u_k is a nonnegative subsolution and $u_{k+1} \leq u_k$. Therefore, $A_{k+1} \leq A_k$.

Question 43. *Prove that Theorem 4.1 is equivalent to the statement: if $A_0 \leq \delta_0$, then $\lim_{k\to\infty} A_k = 0$.*

Question 44. *Prove that*

$$\|u_{k+1}\|_{L^p(B_{r_{k+1}})} \leq C 2^k \|u_{k+1}\|_{L^2(B_{r_k})},$$

where $1/p = 1/2 - 1/d$.

Hint: combine the Cacciopoli inequality with the Sobolev inequality.

Question 45. *Prove that*

$$\|u_{k+1}\|_{L^2(B_{r_{k+1}})} \leq C 2^k \|u_{k+1}\|_{L^2(B_{r_k})} |\{u_{k+1} > 0\} \cap B_{r_{k+1}}|^{2/d}.$$

Hint: combine the previous question with Hölder's inequality.

Question 46. *Prove that*

$$A_{k+1} \leq C 2^{k+4k/d} A_k^{1+4/d}.$$

And this recurrent relation implies that if $A_0 < \delta_0$ for some small δ_0, then $\lim_{k\to\infty} A_k = 0$.

Hint: combine the previous question with Chevischev's inequality.

After solving the questions above, we are done with the proof of Theorem 4.1.

Question 47. *Let $u : B_2 \to \mathbb{R}$ be a nonnegative supersolution (i.e. $-u$ is a subsolution). Prove that there is a constant $\varepsilon_0 > 0$ so that if*

$$|\{x \in B_2 : u(x) \geq 1\}| \geq (1 - \varepsilon_0)|B_2|,$$

then $u(x) \geq 1/2$ almost everywhere in B_1.

Question 48. *For any constants C, $\delta_0 > 0$ and $\delta_1 > 0$, prove that there exists an $\varepsilon > 0$ (depending only on these constants and dimension) so that the following statement is true.*

If $u : B_1 \to [0, 1]$ is such that $\|u\|_{H^1(B_1)} \leq C$, $|\{u = 0\}| \geq \delta_0$ and $|\{u = 1\}| \geq \delta_1$, then $|\{0 < u(x) < 1\}| > \varepsilon$.

Hint. Assume the opposite. There would be a sequence of functions with certain properties. Use Rellich-Kondrachov theorem to obtain a subsequence which converges in L^2. What would the limit function be?

Question 49. *Let $u : B_2 \to \mathbb{R}$ be a nonnegative supersolution. Assume that $|\{x \in B_2 : u(x) \geq 1\}| \geq \delta$. Let*

$$\delta_k := |\{x \in B_{3/2} : u(x) < 2^{-k}\}|.$$

Prove that $\delta_k \to 0$ as $k \to \infty$. Moreover, an upper bound for δ_k depends on δ, k, the ellipticity constants and dimension only (it may not be explicit though).

Question 50. *Let $u : B_2 \to \mathbb{R}$ be a nonnegative supersolution. Assume that $|\{x \in B_2 : u(x) \geq 1\}| \geq \delta$. Prove that*

$$\text{ess-inf}_{B_1} u \geq \theta,$$

for some $\theta > 0$ which depends on δ, the ellipticity constants and dimension only.

Hint. Combine questions 47 and 49.

Question 51. *Let $u : B_2 \to [0, 1]$ be a solution. Prove that*

$$\operatorname*{osc}_{B_1} u := \left(\text{ess-sup}_{B_1} u - \text{ess-inf}_{B_1} u\right) \leq (1 - \theta),$$

for some $\theta > 0$ depending only on dimension and the ellipticity constants.

Hint: either $|\{x \in B_2 : u(x) \geq 1/2\}| \geq |B_2|/2$ or $|\{x \in B_2 : u(x) \leq 1/2\}| \geq |B_2|/2$.

Question 52. *Let $u : B_2 \to \mathbb{R}$ be a solution. Then,*

$$\|u\|_{C^\alpha(B_1)} \leq C\|u\|_{L^2(B_2)}.$$

for some constants C and $\alpha > 0$ depending on dimension and the ellipticity constants only.

Hint: at any point $x_0 \in B_{1/2}$, iterate the oscillation gain from the previous question.

Question 53. *Let $u : \Omega \to \mathbb{R}$ be a nonnegative supersolution, and $R > 0$, $x_0 \in \Omega$ be such that*

$$B_{2R}(x_0) \subseteq \Omega.$$

If $r < R$ and $\inf_{B_r(x_0)} u \geq a$, prove that

$$\inf_{B_R(x_0)} u \geq c \left(\frac{r}{R}\right)^q a, \tag{4.2}$$

for some $c, q > 0$ universal.

Question 54. *Let $u : B_2 \to \mathbb{R}$ be a nonnegative solution. Our goal is to prove the Harnack inequality,*

$$\sup_{B_{1/4}} u \leq Cu(0), \tag{4.3}$$

for some constant C universal. We will do so by showing that if the ratio $\sup\limits_{B_{1/4}} u/u(0)$ was sufficiently large, then we could construct a sequence of points $\{x_k\}_{k=1}^{\infty} \subseteq B_{1/2}$ such that $u(x_k) \to \infty$, contradicting the boundedness of u in B_1.

1. *Prove that for any $x_k \in B_{1/2}$ and r_k sufficiently small that*

$$u(0) \geq cr_k^q \inf_{B_{r_k}(x_k)} . \tag{4.4}$$

2. *Taking $x_{k+1} = \sup\limits_{B_{r_k}(x_k)} u$, prove that*

$$u(x_{k+1}) \geq \frac{u(x_k) - c^{-1} r_k^{-q} u(0)}{1-\theta}. \tag{4.5}$$

3. *Show that if $\dfrac{\sup_{B_{1/4}} u}{u(0)}$ was sufficiently large, then you could choose r_k so that*

$$u(x_{k+1}) \geq \beta u(x_k), \tag{4.6}$$

   for some $\beta > 1$, and that

$$\sum_{k=1}^{\infty} r_k \leq 1/2. \tag{4.7}$$

   Complete the proof.

4. *Prove that*

$$\sup_{B_{1/8}} u \leq C \inf_{B_{1/8}} u. \tag{4.8}$$

5 Week 6

6 More about uniformly elliptic equations with rough coefficients

The first question suggests that perhaps we do not always need to worry about weird functions.

Question 55. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set, $u_n : \Omega \to \mathbb{R}$, $f_n : \Omega \to \mathbb{R}$ and $a^n : \Omega \to \mathbb{R}^{d\times d}$ be sequences so that*

- *For each $n = 1, 2, 3, \ldots$,*

$$\partial_i [a_{ij}^n(x) \partial_j u_n] = f_n \qquad \text{in } \Omega.$$

- *The coefficients a_{ij}^n are uniformly elliptic, with constants uniform in n. Moreover $a_{ij}^n \to a_{ij}$ almost everywhere in Ω.*
- *$f_n \to f$ in $H^{-1}(\Omega)$.*
- *$u_n \to u$ in $H^1(\Omega)$.*

Then,

$$\partial_i [a_{ij}(x) \partial_j u] = f \qquad \text{in } \Omega. \tag{6.1}$$

Conversely, if we have a solution to (6.1), there are sequences u_n, f_n and a^n of C^{∞} functions as above.

Question 56. *Let $u \in H^1(B_1^+)$ be a solution of the equation*

$$\partial_i[a_{ij}(x)\partial_j u] = 0 \qquad \text{in } B_1^+,$$

where $\{a_{ij}\} : B_1^+ \to \mathbb{R}^{d\times d}$ are uniformly elliptic measurable coefficients. Assume that the trace of u on $B_1 \cap \{x_n = 0\}$ is zero. Consider the reflection:

$$\begin{aligned} u(x', -x_n) &= -u(x', x_n) \qquad \text{for } (x', x_n) \in B_1^+, \\ a_{ij}(x', -x_n) &= a_{ij}(x', x_n). \end{aligned}$$

Prove that this extended function $u : B_1 \to \mathbb{R}$ (yes, I still call it u) satisfies the equation

$$\partial_i[a_{ij}(x)\partial_j u] = 0 \qquad \text{in } B_1.$$

Hint. Do some surgery. At some point you will need to notice that $\nabla u(x', x_n) = \nabla u(x', -x_n)$.

Question 57. *For any $\alpha > 0$, prove that there exists a solution to a uniformly elliptic equation in B_1 which is not C^α at the origin (of course, the uniform ellipticity constants will depend on α).*

Question 58. *Let $f \in L^p(B_1)$ for some $p > d/2$. Let u be a solution of*

$$\begin{aligned} \partial_i[a_{ij}(x)\partial_j u] &= f \text{ in } B_1, \\ u &= 0 \text{ on } \partial B_1. \end{aligned}$$

Then

$$\|u\|_{L^\infty(B_1)} \le C\|f\|_{L^p(B_1)}.$$

Moreover, u is Hölder continuous in $\overline{B_1}$ with a norm depending on ellipticity, dimension and $\|f\|_{L^p}$ only.

Hint. Unfortunatelly, to solve this question you might have to redo a significant part of last week's homework. I don't know if there is a smarter way. Do not use LSW theory here since we will need it for the next question (which is literally LSW).

Question 59. *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain. Here $d \ge 3$. Let us consider the operator $S : L^2(\Omega) \to H_0^1(\Omega)$ defined as $Sf := u$ where*

$$\begin{aligned} \partial_i[a_{ij}(x)\partial_j u] &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

Here a_{ij} are symmetric uniformly elliptic coefficients as usual. Prove that there exists a function $G : \Omega\times\Omega \to \mathbb{R}$ such that

$$Sf(x) = \int_\Omega G(x,y)f(y)\,\mathrm{d}y.$$

Needless to say, this function G is the Green function. Moreover,

(a) For every fixed $x \in \Omega$, $G(x,\cdot) \in L^q(\Omega)$ for every $q \in [1, d/(d-2))$.

Hint. *This is the starting point. Use the previous question here.*

(b) The map $x \to G(x,\cdot)$ is continuous from Ω to $L^q(\Omega)$.

(c) We have $G(x,y) = G(y,x)$ and $G \ge 0$.

(d) The function G satisfies the equation

$$\partial_{x_j}[a_{ij}(x)\partial_{x_j} G(x,y)] = 0 \qquad \text{for } (x,y) \in \Omega\times\Omega \setminus \ \{x = y\}.$$

(e) For every fixed $x \in \Omega$, $\nabla_y G(x,\cdot) \in L^1(\Omega)$.

(f) For any C^1 function φ which vanishes on $\partial\Omega$, the following identity holds

$$\varphi(x) = \int_\Omega a_{ij}(y)\partial_i\varphi(y)\partial_{y_i}G(x,y)\,\mathrm{d}y.$$

(g) Prove that

$$\int_{B_{2r}(x)\setminus B_r(x)} |\nabla_y G(x,y)|^2\,\mathrm{d}y \gtrsim r^{2-d}.$$

Hint. *Use part (f) with a well chosen φ.*

(h) There is a constant C (depending on the uniform ellipticity assumption only) such that for every $r > 0$,

$$\sup\{G(x,y) : y \in B_{2r}(x) \setminus B_r(x)\} \le C \inf\{G(x,y) : y \in B_{2r}(x) \setminus B_r(x)\}.$$

Provided that $B_{3r}(x) \subset \Omega$.

Hint. *Use Harnack's inequality.*

(i) Let $m = \inf\{G(x,y) : y \in B_{2r}(x) \setminus B_r(x)\}$. Assume $B_{4r}(x) \subset \Omega$. Prove that

$$\int_{B_{2r}(x)\setminus B_r(x)} |\nabla_y G(x,y)|^2\,\mathrm{d}y \lesssim m^2 r^{d-2}.$$

Hint. *Use Cacciopoli's inequality in a slightly larger ring.*

(j) Prove that if $|x-y| < \operatorname{dist}(x,\partial\Omega)/2$,

$$G(x,y) \gtrsim |x-y|^{2-d}.$$

Hint. *Combine (g) and (i).*

(k) Prove the other inequality

$$G(x,y) \lesssim |x-y|^{2-d}.$$

*(lack of)***Hint.** *You're on your own here.*

6.1 Back to the obstacle problem

Question 60. *Let $w : B_R \to \mathbb{R}$ be a solution to the obstacle problem $\min(w, 1 - \Delta w) = 0$. Assume that $w(x) = 0$ for some $x \in B_1$ and $R > 2$. Prove that*

$$\|w\|_{C^{1,1}(B_{R/2})} \le C,$$

for some universal constant C.

Question 61. *We define the* width *of a set $\Lambda \subset \mathbb{R}^d$ as*

$$\operatorname{width}(\Lambda) := \inf_{|e|=1} \operatorname{osc}\{x \cdot e : x \in \Lambda\}.$$

Here $\operatorname{osc} A = \sup A - \inf A$.

Prove that if $u_n : B_2 \to \mathbb{R}$ is a sequence of continuous functions which converges uniformly to u in B_2, then

$$\limsup_{n\to\infty} \operatorname{width}(\{u_n = 0\} \cap \overline{B}_1) \le \operatorname{width}(\{u = 0\} \cap \overline{B}_1).$$