On the superharmonicity of the first eigenfunction of the fractional Laplacian for certain exponents

Moritz Kassmann and Luis Silvestre

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Abstract

We prove that the first eigenfunction of the fractional Laplacian $(-\Delta)^{\alpha/2}$ in an arbitrary domain is superharmonic provided that $2/\alpha$ is an integer.

1 Introduction

Let us consider the first eigenfunction of the fractional Laplacian in an interval of the real line. We can assume this interval is $(-1,1)$. This function φ is nonnegative and for some $\lambda > 0$ satisfies the equation

$$\begin{aligned} (-\partial_{xx})^{\alpha/2}\varphi &= \lambda\varphi \qquad \text{in } (-1,1), \\ \varphi &= 0 \qquad \text{in } \mathbb{R}\setminus(-1,1). \end{aligned} \tag{1.1}$$

A remarkable open problem is to establish whether φ is a concave function in $(-1,1)$ for any value of $\alpha \in (0,2)$.

We will analyze the n-dimensional version. Here $\Omega \subset \mathbb{R}^n$ is an arbitrary open domain with a $C^{1,1}$ boundary.

$$\begin{aligned} (-\Delta)^{\alpha/2}\varphi &= \lambda\varphi \qquad \text{in } \Omega, \\ \varphi &= 0 \qquad \text{in } \mathbb{R}^n\setminus\Omega. \end{aligned} \tag{1.2}$$

We prove the following result

Theorem 1.1. *If $\alpha = 2/k$ for some integer $k > 0$ and $\Omega \subset \mathbb{R}^n$ is a bounded open set with a $C^{1,1}$ boundary, then the first eigenfunction φ of $(-\Delta)^{\alpha/2}$ is superharmonic.*

Clearly Theorem 1.1 implies the concavity of φ in the one dimensional case.

The purpose of this note is to prove Theorem 1.1. An alternative proof is also given in [1].

The $C^{1,1}$ regularity assumption of the boundary does not seem to play an essential role. It is convenient for some steps in the proof to be well defined. It should be possible to refine the proof to include domains with a rougher boundary. In particular, a straightforward modification of the methods below would extend to domains Ω which satisfy the exterior ball condition.

2 preliminaries

In this section we collect some standard results that we use in the main theorem.

The first proposition says that the first eigenfunction φ, which satisfies (1.2), is smooth in the interior of Ω and $C^{\alpha/2}$ across the boundary $\partial\Omega$.

Proposition 2.1. *The first eigenfunction φ is $C^\infty(\Omega)$ and $C^{\alpha/2}(\mathbb{R}^n)$.*

Proof. The proof of the interior smoothness is very standard. The precise Hölder estimate on the boundary follows by comparing it to a barrier function. See [3]. □

The following proposition is very classical. A proof can be found for example in [2]

Proposition 2.2. *The fractional Laplacian has the integral representation*

$$(-\Delta)^{\alpha/2}\varphi(x) = c\int \frac{u(x)-u(x+y)}{|y|^{n+\alpha}}\,\mathrm{d}y. \tag{2.1}$$

The constant c depends on α and n.

3 Proof of Theorem 1.1

The following two propositions follow by applying the formula (2.1) to the nonnegative function φ at the points in $\mathbb{R}^n \setminus \Omega$, where it vanishes.

Proposition 3.1. *$(-\Delta)^{\alpha/2}u$ is an L^1 function, C^∞ away from $\partial\Omega$, and*

$$(-\Delta)^{\alpha/2}\varphi - \lambda\varphi \le 0 \qquad \text{in } \mathbb{R}^n. \tag{3.1}$$

Moreover,

$$0 \ge (-\Delta)^{\alpha/2}\varphi(x) \ge -C\min\left((|x|-1)^{-\alpha/2}, (|x|-1)^{-\alpha-n}\right) \quad \text{in } \mathbb{R}^n\setminus\Omega.$$

Proof. From Proposition 2, the left hand side is a C^∞ function outside of $\partial\Omega$. Since $\partial\Omega$ is a $C^{1,1}$ surface, from the regularity result in [3], $(-\Delta)^{\alpha/2}u$ is an L^1 function bounded by $C\mathrm{dist}(x,\partial\Omega)^{-\alpha/2}$ for some constant C.

We know that the left hand side is zero inside Ω. It is also negative in $\mathbb{R}^n\setminus\Omega$ since at those points the integrand in Proposition 2.2 has a sign.

The last inequality follows immediately from applying the integral formula of Proposition 2.2. □

Proof of Theorem 1.1. We write $g_j = (-\Delta)^{j\alpha/2}$. The purpose of the proof is to show that $g_k \ge 0$ in Ω.

By definition $g_0 = \varphi$ and $g_1 = \lambda\varphi$ in Ω. On the other hand, from Proposition 3.1, $g_1 < 0$ in $\mathbb{R}^n\setminus\Omega$. We write $g_1 = \lambda\varphi + h$, where h is a non positive function supported in $\mathbb{R}^n\setminus\Omega$ by proposition 3.1.

Applying $(-\Delta)^{\alpha/2}g_1 = (-\Delta)^{\alpha/2}(\lambda\varphi+h)$, we obtain $g_2 = \lambda^2\varphi+\lambda h+(-\Delta)^{\alpha/2}h$. Moreoever, iterating this process we obtain

$$g_r = \lambda^r\varphi + \sum_{j=0}^{r-1}\lambda^{r-j-1}(-\Delta)^{j\alpha/2}h$$

In particular

$$g_k = \lambda^k\varphi + \sum_{j=0}^{k-1}\lambda^{k-j-1}(-\Delta)^{j\alpha/2}h$$

Since h is a non positive function such that $h=0$ in Ω, then $(-\Delta)^s h > 0$ in Ω for any $s\in(0,1)$. Therefore $-\Delta\varphi = g_k$ is given by the sum of all positive terms and it is positive in Ω. □

References

[1] Rodrigo Banuelos and Dante DeBlassie. On the first eigenfunction of the symmetric stable process in a bounded lipschitz domain. *arXiv preprint arXiv:1310.7869*, 2013.

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[3] Xavier Ros-Oton and Joaquim Serra. The dirichlet problem for the fractional laplacian: regularity up to the boundary. *Journal de Mathématiques Pures et Appliquées*, 101(3):275–302, 2014.