On the Hamilton-Jacobi equation with critical diffusion

Luis Silvestre

University of Chicago

March 30, 2010

Introduction

The Hamilton-Jacobi equation with diffusion

References

Advection-diffusion equation.

Differentiating Hamiton-Jacobi

References

Weak solutions

The proof

Generalities

Improvement of oscillation Lemma

Pointwise estimate from measure estimate

The dirty details

Hamilton Jacobi equation

The Hamilton-Jacobi equation

$$u_t + H(\nabla u) = 0$$

appears in deterministic control problems (convex case) or deterministic games (nonconvex case) .

Hamilton Jacobi equation

The Hamilton-Jacobi equation with diffusion

$$u_t + H(\nabla u) - \triangle u = 0$$

appears in control problems (convex case) or games (nonconvex case) with Brownian diffusion .

Hamilton Jacobi equation

The Hamilton-Jacobi equation with fractional diffusion

$$u_t + H(\nabla u) + (-\triangle)^s u = 0$$

appears in control problems (convex case) or games (nonconvex case) with α-stable diffusion.

sub- and super-critical regime

$$u_t + H(\nabla u) + (-\triangle)^s u = 0$$

- Subcritical case $s > 1/2$: smooth solutions can be obtained by a fixed point approach.
- Supercritical case $s < 1/2$: there are viscosity solutions that are Lipschitz but not C^1.
- Critical case $s = 1/2$: All terms of the equation are of order one. No perturbative methods apply.
  The subject of this talk: the solutions are $C^{1,\alpha}$ (classical) in the critical case.

sub- and super-critical regime

$$u_t + H(\nabla u) + (-\triangle)^s u = 0$$

- Subcritical case $s > 1/2$: smooth solutions can be obtained by a fixed point approach.
- Supercritical case $s < 1/2$: there are viscosity solutions that are Lipschitz but not C^1.
- Critical case $s = 1/2$: All terms of the equation are of order one. No perturbative methods apply.
  The subject of this talk: the solutions are $C^{1,\alpha}$ (classical) in the critical case.

sub- and super-critical regime

$$u_t + H(\nabla u) + (-\triangle)^s u = 0$$

- Subcritical case $s > 1/2$: smooth solutions can be obtained by a fixed point approach.
- Supercritical case $s < 1/2$: there are viscosity solutions that are Lipschitz but not C^1.
- Critical case $s = 1/2$: All terms of the equation are of order one. No perturbative methods apply.
  The subject of this talk: the solutions are $C^{1,\alpha}$ (classical) in the critical case.

sub- and super-critical regime

$$u_t + H(\nabla u) + (-\triangle)^s u = 0$$

- Subcritical case $s > 1/2$: smooth solutions can be obtained by a fixed point approach.
- Supercritical case $s < 1/2$: there are viscosity solutions that are Lipschitz but not C^1.
- Critical case $s = 1/2$: All terms of the equation are of order one. No perturbative methods apply.

  The subject of this talk: the solutions are $C^{1,\alpha}$ (classical) in the critical case.

sub- and super-critical regime

$$u_t + H(\nabla u) + (-\triangle)^s u = 0$$

- Subcritical case $s > 1/2$: smooth solutions can be obtained by a fixed point approach.
- Supercritical case $s < 1/2$: there are viscosity solutions that are Lipschitz but not C^1.
- Critical case $s = 1/2$: All terms of the equation are of order one. No perturbative methods apply.
  The subject of this talk: the solutions are $C^{1,\alpha}$ (classical) in the critical case.

Earlier references

The equations $u_t + H(\nabla u) + (-\triangle)^s u = 0$ has previously been considered in the following works:

- J. Droniou,T. Gallouët, and J. Vovelle. JEE 2003: One dimensional case. The regularity for $s = 1/2$ is left open.
- C. Imbert. JDE 2005: Case $s > 1/2$.
- C. Imbert and J. Droniou. ARMA 2006: Lipschitz viscosity solutions for $s \in (0, 1)$.
- G. Karch and W. Woyczynski. TAMS 2008: Long time assymptotics.
- Several authors in recent years: Biler, Funaki, Woyczynski, Jourdain, Méléard, Droniou, Imbert, Caffarelli, Vasseur, Czubak, Chan, Kiselev, Nazarov, Shterenberg: Conservation laws with fractional diffusion.

More general result

The equation $u_t + H(\nabla u) + (-\triangle)^s u = 0$ can be written as

$$\inf_i \sup_j b_{ij} \cdot \nabla u + (-\triangle)^s u = 0$$

A more general version is

$$\inf_i \sup_j b_{ij} \cdot \nabla u + \int \frac{u(x) - u(x+y)}{|y|^{n+2s}} a_{ij}(y) \, \mathrm{d}y = 0$$

where a_{ij} satisfies $\lambda \leq a_{ij} \leq \Lambda$ and $a_{ij}(y) = a_{ij}(-y)$.
The results apply to the most general equation, but in order to keep the presentation simpler, we concentrate in the former.

More general result

The equation $u_t + H(\nabla u) + (-\triangle)^s u = 0$ can be written as

$$\inf_i \sup_j b_{ij} \cdot \nabla u + (-\triangle)^s u = 0$$

A more general version is

$$\inf_i \sup_j b_{ij} \cdot \nabla u + \int \frac{u(x) - u(x+y)}{|y|^{n+2s}} a_{ij}(y) \, \mathrm{d}y = 0$$

where a_{ij} satisfies $\lambda \leq a_{ij} \leq \Lambda$ and $a_{ij}(y) = a_{ij}(-y)$.
The results apply to the most general equation, but in order to keep the presentation simpler, we concentrate in the former.

Strategy for $C^{1,\alpha}$ regularity

We differentiate the equation

$$u_t + H(\nabla u) + (-\triangle)^{1/2} u = 0$$

so that if $v = \partial_e u$,

$$v_t + H'(\nabla u) \cdot \nabla v + (-\triangle)^{1/2} v = 0$$

Strategy for $C^{1,\alpha}$ regularity

We differentiate the equation

$$u_t + H(\nabla u) + (-\triangle)^{1/2} u = 0$$

so that if $v = \partial_e u$,

$$v_t + H'(\nabla u) \cdot \nabla v + (-\triangle)^{1/2} v = 0$$

We assume H to be (locally) Lipschitz.

Strategy for $C^{1,\alpha}$ regularity

We differentiate the equation

$$u_t + H(\nabla u) + (-\triangle)^{1/2} u = 0$$

so that if $v = \partial_e u$,

$$v_t + w \cdot \nabla v + (-\triangle)^{1/2} v = 0$$

For some vector field w in L^∞.

no modulus of continuity can be assumed a priori for w

Advection-(fractional)diffusion equation

The regularity of the critical Hamilton-Jacobi equation follows from the following theorem.

Theorem (S.)

If v solves

$$v_t + w \cdot \nabla v + (-\triangle)^{1/2} v = 0$$

for an arbitrary bounded vector field w, then v becomes immediately Hölder continuous.

This is the important result to prove.

Related results

Theorem (Caffarelli and Vasseur. To appear in Annals of Math.)

If v solves

$$v_t + w \cdot \nabla v + (-\triangle)^{1/2} v = 0$$

for an arbitrary BMO divergence free vector field w, then v becomes immediately Hölder continuous.

Proof.

Extend the equation to one more dimension, rewrite the problem as a local PDE and reproduce De Giorgi-Nash-Moser theorem. □

This result implies the well posedness of the quasi-geostrophic equation. There is another proof given by Kiselev and Nazarov.

Essential difference with our result: divergence structure.

weak solutions

If div $w = 0$, the equation is defined in the weak sense integrating agains a smooth test function φ.

$$\iint (v_t + w \cdot \nabla v + (-\triangle)^{1/2} v)\varphi \, \mathrm{d}x \, \mathrm{d}t = 0$$

What if div $w \neq 0$? All we can say from the equation
$v_t + w \cdot \nabla v + (-\triangle)^{1/2} v = 0$
are the following two inequalities:

$$v_t + A|\nabla v| + (-\triangle)^{1/2} v \geq 0$$

$$v_t - A|\nabla v| + (-\triangle)^{1/2} v \leq 0$$

which are well defined in the viscosity sense.

weak solutions

If div $w = 0$, the equation is defined in the weak sense integrating agains a smooth test function φ.

$$\iint v(-\varphi_t - w \cdot \nabla\varphi + (-\triangle)^{1/2}\varphi)\, \mathrm{d}x\, \mathrm{d}t = 0$$

What if div $w \neq 0$? All we can say from the equation
$v_t + w \cdot \nabla v + (-\triangle)^{1/2} v = 0$
are the following two inequalities:

$$v_t + A|\nabla v| + (-\triangle)^{1/2} v \geq 0$$
$$v_t - A|\nabla v| + (-\triangle)^{1/2} v \leq 0$$

which are well defined in the viscosity sense.

weak solutions

If $\operatorname{div} w = 0$, the equation is defined in the weak sense integrating agains a smooth test function φ.

$$\iint v(-\varphi_t - w \cdot \nabla\varphi + (-\triangle)^{1/2}\varphi)\, \mathrm{d}x\, \mathrm{d}t = 0$$

What if $\operatorname{div} w \neq 0$? All we can say from the equation
$v_t + w \cdot \nabla v + (-\triangle)^{1/2} v = 0$
are the following two inequalities:

$$v_t + A|\nabla v| + (-\triangle)^{1/2} v \geq 0$$
$$v_t - A|\nabla v| + (-\triangle)^{1/2} v \leq 0$$

which are well defined in the viscosity sense.

weak solutions

If div $w = 0$, the equation is defined in the weak sense integrating agains a smooth test function φ.

$$\iint v(-\varphi_t - w \cdot \nabla\varphi + (-\triangle)^{1/2}\varphi) \, \mathrm{d}x \, \mathrm{d}t = 0$$

What if div $w \neq 0$? All we can say from the equation
$v_t + w \cdot \nabla v + (-\triangle)^{1/2} v = 0$
are the following two inequalities:

$$v_t + A|\nabla v| + (-\triangle)^{1/2} v \geq 0$$
$$v_t - A|\nabla v| + (-\triangle)^{1/2} v \leq 0$$

which are well defined in the viscosity sense.

General strategy for proving Hölder continuity

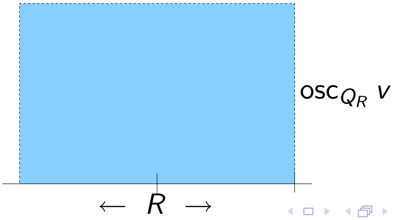
First prove an improvement of oscillation lemma

$$\operatorname*{osc}_{Q_{\lambda R}} v \leq (1-\theta) \operatorname*{osc}_{Q_R} v \qquad Q_R := B_R \times [-R, 0]$$

and then iterate it to obtain

$$\operatorname*{osc}_{Q_{\lambda^k R}} v \leq (1-\theta)^k \operatorname*{osc}_{Q_R} v$$

which imples a C^α modulus of continuity for $\alpha = \frac{\log(1-\theta)}{\log(\lambda)}$.



General strategy for proving Hölder continuity

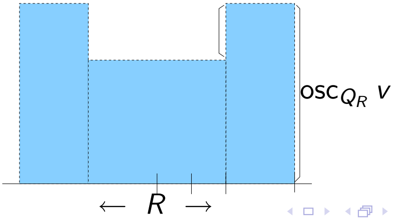
First prove an improvement of oscillation lemma

$$\operatorname*{osc}_{Q_{\lambda R}} v \leq (1-\theta) \operatorname*{osc}_{Q_R} v \qquad Q_R := B_R \times [-R, 0]$$

and then iterate it to obtain

$$\operatorname*{osc}_{Q_{\lambda^k R}} v \leq (1-\theta)^k \operatorname*{osc}_{Q_R} v$$

which imples a C^α modulus of continuity for $\alpha = \frac{\log(1-\theta)}{\log(\lambda)}$.



General strategy for proving Hölder continuity

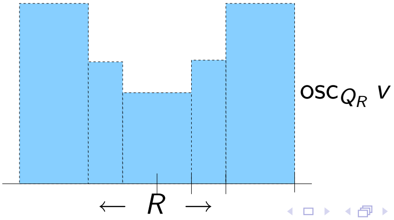
First prove an improvement of oscillation lemma

$$\operatorname*{osc}_{Q_{\lambda R}} v \leq (1-\theta) \operatorname*{osc}_{Q_R} v \qquad Q_R := B_R \times [-R, 0]$$

and then iterate it to obtain

$$\operatorname*{osc}_{Q_{\lambda^k R}} v \leq (1-\theta)^k \operatorname*{osc}_{Q_R} v$$

which imples a C^α modulus of continuity for $\alpha = \frac{\log(1-\theta)}{\log(\lambda)}$.



General strategy for proving Hölder continuity

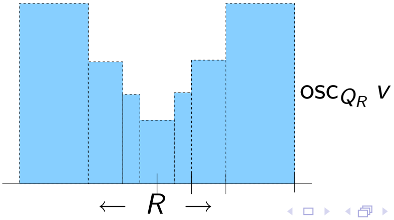
First prove an improvement of oscillation lemma

$$\operatorname*{osc}_{Q_{\lambda R}} v \leq (1-\theta) \operatorname*{osc}_{Q_R} v \qquad Q_R := B_R \times [-R, 0]$$

and then iterate it to obtain

$$\operatorname*{osc}_{Q_{\lambda^k R}} v \leq (1-\theta)^k \operatorname*{osc}_{Q_R} v$$

which imples a C^α modulus of continuity for $\alpha = \frac{\log(1-\theta)}{\log(\lambda)}$.



General strategy for proving Hölder continuity

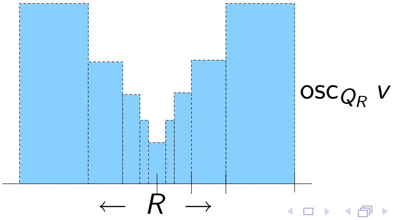
First prove an improvement of oscillation lemma

$$\operatorname*{osc}_{Q_{\lambda R}} v \leq (1-\theta) \operatorname*{osc}_{Q_R} v \qquad Q_R := B_R \times [-R, 0]$$

and then iterate it to obtain

$$\operatorname*{osc}_{Q_{\lambda^k R}} v \leq (1-\theta)^k \operatorname*{osc}_{Q_R} v$$

which imples a C^α modulus of continuity for $\alpha = \frac{\log(1-\theta)}{\log(\lambda)}$.



General strategy for proving Hölder continuity

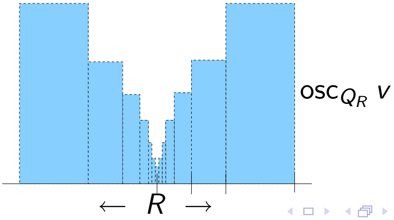
First prove an improvement of oscillation lemma

$$\operatorname*{osc}_{Q_{\lambda R}} v \leq (1-\theta) \operatorname*{osc}_{Q_R} v \qquad Q_R := B_R \times [-R, 0]$$

and then iterate it to obtain

$$\operatorname*{osc}_{Q_{\lambda^k R}} v \leq (1-\theta)^k \operatorname*{osc}_{Q_R} v$$

which imples a C^α modulus of continuity for $\alpha = \frac{\log(1-\theta)}{\log(\lambda)}$.



General strategy for proving Hölder continuity

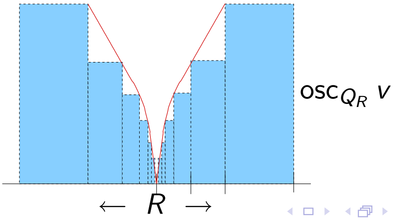
First prove an improvement of oscillation lemma

$$\operatorname*{osc}_{Q_{\lambda R}} v \leq (1-\theta) \operatorname*{osc}_{Q_R} v \qquad Q_R := B_R \times [-R, 0]$$

and then iterate it to obtain

$$\operatorname*{osc}_{Q_{\lambda^k R}} v \leq (1-\theta)^k \operatorname*{osc}_{Q_R} v$$

which imples a C^α modulus of continuity for $\alpha = \frac{\log(1-\theta)}{\log(\lambda)}$.

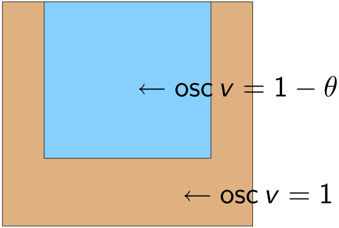


Improvement of oscillation lemma

If $\operatorname{osc}_{Q_R} v \leq 1$,

$$v_t + A|\nabla v| + (-\triangle)^{1/2} v \geq 0$$
$$v_t - A|\nabla v| + (-\triangle)^{1/2} v \leq 0$$

then $\operatorname{osc}_{Q_1} v \leq 1 - \theta$.

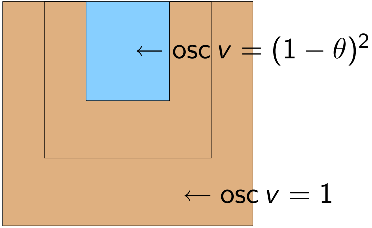


Improvement of oscillation lemma

If $\operatorname{osc}_{Q_R} v \leq 1$,

$$v_t + A|\nabla v| + (-\triangle)^{1/2} v \geq 0$$
$$v_t - A|\nabla v| + (-\triangle)^{1/2} v \leq 0$$

then $\operatorname{osc}_{Q_1} v \leq 1 - \theta$.

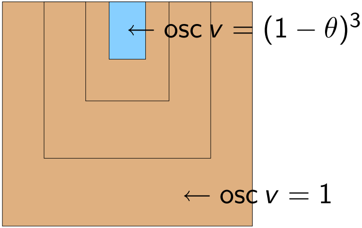


Improvement of oscillation lemma

If $\operatorname{osc}_{Q_R} v \leq 1$,

$$v_t + A|\nabla v| + (-\triangle)^{1/2} v \geq 0$$
$$v_t - A|\nabla v| + (-\triangle)^{1/2} v \leq 0$$

then $\operatorname{osc}_{Q_1} v \leq 1 - \theta$.

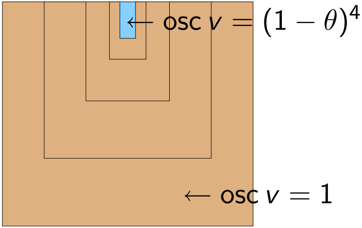


Improvement of oscillation lemma

If $\operatorname{osc}_{Q_R} v \leq 1$,

$$v_t + A|\nabla v| + (-\triangle)^{1/2} v \geq 0$$
$$v_t - A|\nabla v| + (-\triangle)^{1/2} v \leq 0$$

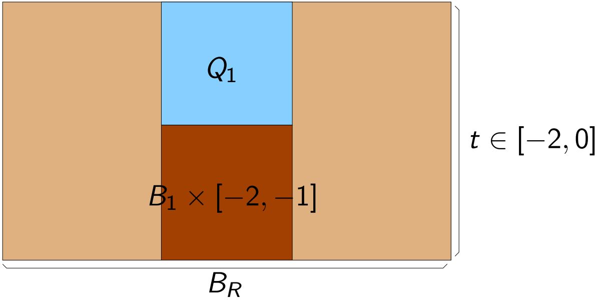
then $\operatorname{osc}_{Q_1} v \leq 1 - \theta$.



Improvement of either the sup or the inf

We prove an improvement of oscillation from $B_R \times [-2,0]$ to $B_1 \times [-1,0]$ $(= Q_1)$.

For the improvement of oscillation we prove that either sup v decreases or inf v increases. We do one or the other depending on wheather $v \geq 0$ or $v \leq 0$ most often (in measure) in $B_1 \times [-2,-1]$.

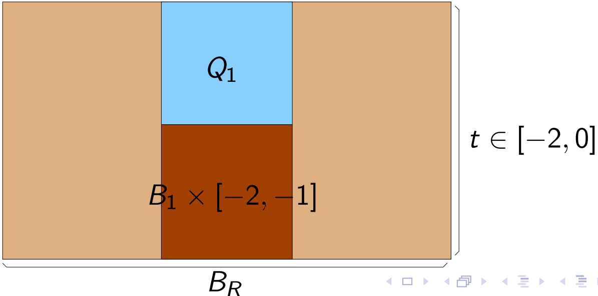


From an estimate in measure

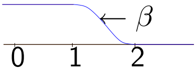
Lemma

If v less than one: $v \leq 1$ *in* $\mathbb{R}^n \times [-2,0]$,
v is often negative: $|\{v \leq 0\} \cap (B_1 \times [-2,-1])| \geq \mu$, *and*
v is a subsolution: $v_t - A|\nabla v| + (-\triangle)^{1/2} v \leq 0$ *in* $B_R \times [-2,0]$

then $v \leq 1 - \theta$ *in* $B_1 \times [-1,0]$.



Construction of a barrier

Let β be the function: 

and $b(x,t) = \beta(|x| - A|t|)$.

The function $w = 1 - mb(x,t)$ is a supersolution of the pure transport part $w_t - A|\nabla w| = 0$ for any constant m.

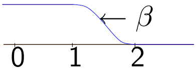
We use the diffusion term to make m variable. Let m solve

$$m(-2) = 0$$
$$m'(t) = c_0|\{x \in [-1,1] : u(x,t) \leq 0\}| - C_1 m(t).$$

We will show that if $c_0 \ll 1$ and $C_1 \gg 1$ then $v \leq 1 - m(t)b(x,t)$ everywhere.

Maximum principle type estimate: There cannot be a first time when $v(x,t) = 1 - m(t)b(x,t)$.

Construction of a barrier

Let β be the function: 

and $b(x,t) = \beta(|x| - A|t|)$.

The function $w = 1 - mb(x,t)$ is a supersolution of the pure transport part $w_t - A|\nabla w| = 0$ for any constant m.

We use the diffusion term to make m variable. Let m solve

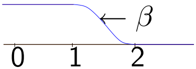
$$m(-2) = 0$$

$$m'(t) = c_0 |\{x \in [-1,1] : u(x,t) \leq 0\}| - C_1 m(t).$$

We will show that if $c_0 \ll 1$ and $C_1 \gg 1$ then $v \leq 1 - m(t)b(x,t)$ everywhere.

Maximum principle type estimate: There cannot be a first time when $v(x,t) = 1 - m(t)b(x,t)$.

Construction of a barrier

Let β be the function: 

and $b(x,t) = \beta(|x| - A|t|)$.

The function $w = 1 - mb(x,t)$ is a supersolution of the pure transport part $w_t - A|\nabla w| = 0$ for any constant m.

We use the diffusion term to make m variable. Let m solve

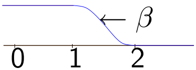
$$m(-2) = 0$$

$$m'(t) = c_0|\{x \in [-1,1] : u(x,t) \le 0\}| - C_1 m(t).$$

We will show that if $c_0 \ll 1$ and $C_1 \gg 1$ then $v \le 1 - m(t)b(x,t)$ everywhere.

Maximum principle type estimate: There cannot be a first time when $v(x,t) = 1 - m(t)b(x,t)$.

Construction of a barrier

Let β be the function: 

and $b(x,t) = \beta(|x| - A|t|)$.

The function $w = 1 - mb(x,t)$ is a supersolution of the pure transport part $w_t - A|\nabla w| = 0$ for any constant m.
We use the diffusion term to make m variable. Let m solve

$$m(-2) = 0$$
$$m'(t) = c_0|\{x \in [-1,1] : u(x,t) \leq 0\}| - C_1 m(t).$$

We will show that if $c_0 \ll 1$ and $C_1 \gg 1$ then $v \leq 1 - m(t)b(x,t)$ everywhere.

Maximum principle type estimate: There cannot be a first time when $v(x,t) = 1 - m(t)b(x,t)$.

Contradiction if v and $1 - mb$ ever touch

If $v(x,t)$ coincides with $w(x,t) = 1 - m(t)b(x,t)$ for the first time at a point (x,t) the following happen:

$$v_t \geq w_t = -m'b - mb_t = -m'b + mA|\nabla b|$$

$$|\nabla v| = |\nabla w| = m|\nabla b|$$

$$(-\triangle)^{1/2} v \geq (-\triangle)^{1/2} w = -m(-\triangle)^{1/2} b \quad \leftarrow \text{can be improved!}$$

Contradiction if v and $1 - mb$ ever touch

If $v(x,t)$ coincides with $w(x,t) = 1 - m(t)b(x,t)$ for the first time at a point (x,t) the following happen:

$$v_t \geq w_t = -m'b - mb_t = -m'b + mA|\nabla b|$$

$$|\nabla v| = |\nabla w| = m|\nabla b|$$

$$(-\triangle)^{1/2} v \geq (-\triangle)^{1/2} w = -m(-\triangle)^{1/2} b \quad \leftarrow \text{can be improved!}$$

Contradiction if v and $1 - mb$ ever touch

If $v(x,t)$ coincides with $w(x,t) = 1 - m(t)b(x,t)$ for the first time at a point (x,t) the following happen:

$$v_t \geq w_t = -m'b - mb_t = -m'b + mA|\nabla b|$$

$$|\nabla v| = |\nabla w| = m|\nabla b|$$

$$(-\triangle)^{1/2} v \geq (-\triangle)^{1/2} w = -m(-\triangle)^{1/2} b \quad \leftarrow \text{can be improved!}$$

Contradiction if v and $1 - mb$ ever touch

If $v(x,t)$ coincides with $w(x,t) = 1 - m(t)b(x,t)$ for the first time at a point (x,t) the following happen:

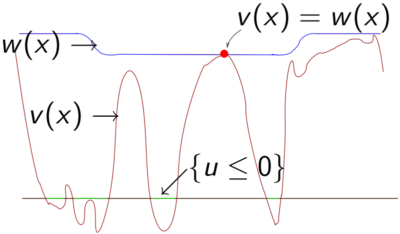
$$v_t \geq w_t = -m'b - mb_t = -m'b + mA|\nabla b|$$

$$|\nabla v| = |\nabla w| = m|\nabla b|$$

$$(-\triangle)^{1/2} v \geq (-\triangle)^{1/2} w = -m(-\triangle)^{1/2} b \quad \leftarrow \text{can be improved!}$$

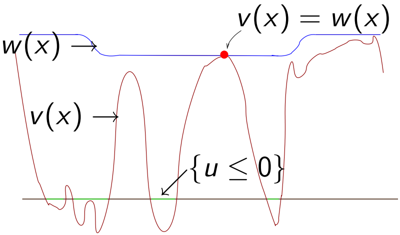
Sharper estimate of $(-\triangle)^{1/2}v$

$$
\begin{aligned}
(-\triangle)^{1/2}v(x) &= c\int \frac{v(x)-v(y)}{|x-y|^{n+1}}\,\mathrm{d}y \\
&\geq c\int \frac{w(x)-w(y)}{|x-y|^{n+1}}\,\mathrm{d}y \\
&\geq (-\triangle)^{1/2}w(x) + c_0|\{u<0\}\cap B_1|
\end{aligned}
$$



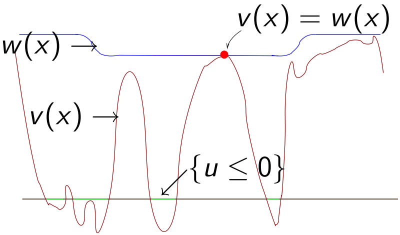
Sharper estimate of $(-\triangle)^{1/2} v$

$$
\begin{aligned}
(-\triangle)^{1/2} v(x) &= c \int \frac{v(x) - v(y)}{|x-y|^{n+1}} \, \mathrm{d}y \\
&\geq c \int \frac{w(x) - w(y)}{|x-y|^{n+1}} \, \mathrm{d}y + c \int_{u(y) \leq 0} \frac{w(y)}{|x-y|^{n+1}} \, \mathrm{d}y \\
&\geq (-\triangle)^{1/2} w(x) + c_0 |\{u < 0\} \cap B_1|
\end{aligned}
$$



Sharper estimate of $(-\triangle)^{1/2}v$

$$
\begin{aligned}
(-\triangle)^{1/2}v(x) &= c\int \frac{v(x)-v(y)}{|x-y|^{n+1}}\,\mathrm{d}y \\
&\geq c\int \frac{w(x)-w(y)}{|x-y|^{n+1}}\,\mathrm{d}y + c\int_{u(y)\leq 0}\frac{w(y)}{|x-y|^{n+1}}\,\mathrm{d}y \\
&\geq (-\triangle)^{1/2}w(x) + c_0|\{u<0\}\cap B_1|
\end{aligned}
$$



Putting all estimates together

$$v_t \geq -m' b + mA|\nabla b|$$

$$|\nabla v| = m|\nabla b|$$

$$(-\triangle)^{1/2} v \geq -m(-\triangle)^{1/2} b(x) + c_0 |\{u < 0\} \cap B_1|$$

Therefore

$$v_t - A|\nabla v| + (-\triangle)^{1/2} v \geq -m' b - m(-\triangle)^{1/2} b(x) + c_0 |\{u < 0\} \cap B_1|$$

Two cases:

1. $b(x,t)$ is small $\Rightarrow (-\triangle)^{1/2} b \leq 0$.
2. $b(x,t)$ is large $\Rightarrow$ we choose C_1 large.

Contradiction with $v_t - A|\nabla v| + (-\triangle)^{1/2} v \leq 0$

Putting all estimates together

$$\begin{aligned} v_t &\geq -m' b + mA|\nabla b| \\ |\nabla v| &= m|\nabla b| \\ (-\triangle)^{1/2} v &\geq -m(-\triangle)^{1/2} b(x) + c_0 |\{u < 0\} \cap B_1| \end{aligned}$$

Therefore

$$v_t - A|\nabla v| + (-\triangle)^{1/2} v \geq C_1 mb - m(-\triangle)^{1/2} b(x)$$

Two cases:

1. $b(x,t)$ is small $\Rightarrow (-\triangle)^{1/2} b \leq 0$.
2. $b(x,t)$ is large $\Rightarrow$ we choose C_1 large.

Contradiction with $v_t - A|\nabla v| + (-\triangle)^{1/2} v \leq 0$

Putting all estimates together

$$
\begin{aligned}
v_t &\geq -m' b + mA|\nabla b| \\
|\nabla v| &= m|\nabla b| \\
(-\triangle)^{1/2} v &\geq -m(-\triangle)^{1/2} b(x) + c_0 |\{u < 0\} \cap B_1|
\end{aligned}
$$

Therefore

$$
v_t - A|\nabla v| + (-\triangle)^{1/2} v \geq C_1 m b - m(-\triangle)^{1/2} b(x)
$$

Two cases:

1. $b(x,t)$ is small $\Rightarrow (-\triangle)^{1/2} b \leq 0$.
2. $b(x,t)$ is large $\Rightarrow$ we choose C_1 large.

Contradiction with $v_t - A|\nabla v| + (-\triangle)^{1/2} v \leq 0$

Putting all estimates together

$$
\begin{aligned}
v_t &\geq -m' b + mA|\nabla b| \\
|\nabla v| &= m|\nabla b| \\
(-\triangle)^{1/2} v &\geq -m(-\triangle)^{1/2} b(x) + c_0 |\{u < 0\} \cap B_1|
\end{aligned}
$$

Therefore

$$v_t - A|\nabla v| + (-\triangle)^{1/2} v \geq C_1 mb - m(-\triangle)^{1/2} b(x)$$

Two cases:

1. $b(x,t)$ is small $\Rightarrow (-\triangle)^{1/2} b \leq 0$.
2. $b(x,t)$ is large $\Rightarrow$ we choose C_1 large.

Contradiction with $v_t - A|\nabla v| + (-\triangle)^{1/2} v \leq 0$.

Putting all estimates together

$$v_t \geq -m' b + mA|\nabla b|$$
$$|\nabla v| = m|\nabla b|$$
$$(-\triangle)^{1/2} v \geq -m(-\triangle)^{1/2} b(x) + c_0 |\{u < 0\} \cap B_1|$$

Therefore

$$v_t - A|\nabla v| + (-\triangle)^{1/2} v \geq C_1 mb - m(-\triangle)^{1/2} b(x)$$

Two cases:

1. $b(x,t)$ is small $\Rightarrow (-\triangle)^{1/2} b \leq 0$.
2. $b(x,t)$ is large $\Rightarrow$ we choose C_1 large.

Contradiction with $v_t - A|\nabla v| + (-\triangle)^{1/2} v \leq 0$.

Putting all estimates together

$$v_t \geq -m' b + mA|\nabla b|$$
$$|\nabla v| = m|\nabla b|$$
$$(-\triangle)^{1/2} v \geq -m(-\triangle)^{1/2} b(x) + c_0 |\{u < 0\} \cap B_1|$$

Therefore

$$v_t - A|\nabla v| + (-\triangle)^{1/2} v \geq C_1 m b - m(-\triangle)^{1/2} b(x)$$

Two cases:

1. $b(x,t)$ is small $\Rightarrow (-\triangle)^{1/2} b \leq 0$.
2. $b(x,t)$ is large $\Rightarrow$ we choose C_1 large.

Contradiction with $v_t - A|\nabla v| + (-\triangle)^{1/2} v \leq 0$.