Uniformly elliptic equations that hold only at points of large gradient.

Luis Silvestre

University of Chicago

Joint work with Cyril Imbert

The Krylov-Safonov theorem

Let u be a solution to

$$a_{ij}(x)\partial_{ij}u = 0 \quad \text{in } B_1.$$

The coefficients a_{ij} are not assumed to be any regular, but just to satisfy the following inequalities pointwise

$$\lambda \mathrm{I} \leq a_{ij}(x) \leq \Lambda \mathrm{I}.$$

The u is Hölder continuous in $B_{1/2}$ with the estimate

$\|u\|_{C^\alpha(B_{1/2})} \leq C\|u\|_{L^\infty(B_1)}$ $\qquad C$ depends on $\lambda, \Lambda,$ and dimension only.

A Harnack inequality also holds.

De Giorgi-Nash-Moser theorem

Let u be a solution to

$$\partial_i a_{ij}(x) \partial_j u = 0 \quad \text{in } B_1.$$

The coefficients a_{ij} are not assumed to be any regular (besides measurable), but just to satisfy the following inequalities pointwise

$$\lambda \mathrm{I} \leq a_{ij}(x) \leq \Lambda \mathrm{I}.$$

The u is Hölder continuous in $B_{1/2}$ with the estimate

$\|u\|_{C^\alpha(B_{1/2})} \leq C \|u\|_{L^2(B_1)}$, $\qquad$ C depends on $\lambda, \Lambda,$ and dimension only.

A Harnack inequality also holds.

One application to fully nonlinear equations

Fully nonlinear equation: $F(D^2u) = 0$ in B_1.

We assume the uniform ellipticity condition: $\lambda \mathrm{I} \leq \frac{\partial F}{\partial X_{ij}}(X) \leq \Lambda \mathrm{I}$ for any symmetric matrix X. Then the solutions are $C^{1,\alpha}$ for some $\alpha > 0$.

Sketch-proof. The directional derivatives u_e, satisfy the linearized equation

$$\frac{\partial F}{\partial X_{ij}}(D^2u(x))\, \partial_{ij} u_e = 0 \quad \text{in } B_1.$$

The Krylov-Safonov theorem proves $u_e \in C^{\alpha}$ for any direction e.

One application to fully nonlinear equations

Fully nonlinear equation: $F(D^2u) = 0$ in B_1.

We assume the uniform ellipticity condition: $\lambda \mathrm{I} \leq \frac{\partial F}{\partial X_{ij}}(X) \leq \Lambda \mathrm{I}$ for any symmetric matrix X. Then the solutions are $C^{1,\alpha}$ for some $\alpha > 0$.

Sketch-proof. The directional derivatives u_e, satisfy the linearized equation

$$a_{ij}(x)\partial_{ij}u_e = 0 \quad \text{in } B_1.$$

The Krylov-Safonov theorem proves $u_e \in C^\alpha$ for any direction e.

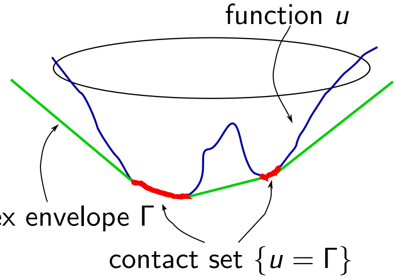
The ABP estimate

A crucial tool in the proof of Krylov-Safonov theorem is the Aleksandrov-Bakelman-Pucci estimate. If we assume

- $a_{ij}(x)\partial_{ij}u \le f$ in B_1 with $\lambda\mathrm{I} \le a_{ij}(x) \le \Lambda\mathrm{I}$.
- $u \ge 0$ on ∂B_1.

Then

$$-\min_{B_1} u \le C\left(\int_{\{u=\Gamma\}} f^n \,\mathrm{d}x\right)^{1/n}$$



Our new result

Theorem (Imbert, S.)

Let u be a solution to

$$a_{ij}(x)\partial_{ij}u = 0 \ \ \textbf{only where } |\nabla \mathbf{u}| \geq \gamma \text{ in } B_1.$$

The coefficients a_{ij} are not assumed to be any regular, but just to satisfy the following inequalities pointwise

$$\lambda \mathrm{I} \leq a_{ij}(x) \leq \Lambda \mathrm{I}.$$

Then u is Hölder continuous in $B_{1/2}$ with the estimate

$$\|u\|_{C^\alpha(B_{1/2})} \leq C\left(\lambda, \Lambda, n, \frac{\gamma}{\|u\|_{L^\infty(B_1)}}\right)\|u\|_{L^\infty(B_1)}.$$

A Harnack inequality also holds.

Related results

Theorem (Davila, Felmer, Quaas. C.R. 2009)

The equation

$$F(\nabla u, D^2 u) + b \cdot \nabla u |\nabla u|^\alpha + cu|u|^\alpha = f,$$

where $|p|^\alpha M^-(X) \leq F(p,X) \leq |p|^\alpha M^+(X)$ *for* $\alpha > -1$*, satisfies (some variation of) the ABP estimate.*

Theorem (Davila, Felmer, Quaas. CVPDE 2010)

The equation

$$F(\nabla u, D^2 u) + b \cdot \nabla u |\nabla u|^\alpha + cu|u|^\alpha = f,$$

where $|p|^\alpha M^-(X) \leq F(p,X) \leq |p|^\alpha M^+(X)$ *for* $\alpha \in (-1,0)$*, has Hölder estimates and satisfies the Harnack inequality.*

Related results 2

Theorem (Delarue, JDE 2010)

$$a_{ij}(x, u(x), \nabla u(x))\partial_{ij} u(x) + F(x, u(x), \nabla u(x)) = 0 \qquad \text{in } B_1,$$

where $\omega(p)\lambda \mathrm{I} \leq a_{ij}(x, u, p) \leq \omega(p)\Lambda \mathrm{I}$ *with* $\omega(p) > 1$ *for* $p > \gamma$*, plus some assumptions on* F*. Then there is Hölder estimates and Harnack inequality.*

From his paper: "what is important is that, at any x, all the eigenvalues of a_{ij} behave in the same way".
The proof is rather involved and based on probabilistic methods (i.e. I don't understand the proof).

Related results 3

Theorem (Cyril Imbert, JDE 2011)

Under similar assumption to our main theorem, he obtained that (some variation of) the Alexandroff-Bakelman-Pucci estimate holds.

The paper argues that Hölder estimates and a Harnack inequality would follow. However, there is a critical flaw in that argument.

Other related results.

- Birindelli and Demengel (preprint) obtained a Harnack inequality for a nonlinear equation in 2D under some conditions that fit into our framework.
- It could be said that the divergence form version of these results was developed in a series of papers by DiBenedetto, Gianazza, Vespri, and others.
- In the opposite case, that the equation is uniformly elliptic for small values of the gradient (and Hessian), the Harnack inequality for sufficiently flat solutions was obtained by Ovidiu Savin (CPDE 2007).
- A related ABP estimate for quasilinear elliptic and parabolic equations of p-Laplacian type was obtained by Roberto Argiolas, Fernando Charro and Ireneo Peral in 2011.

General strategy for proving Hölder continuity

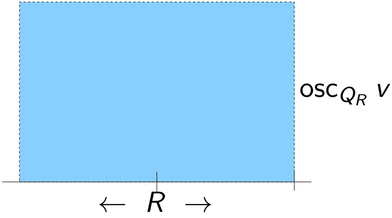
First prove an improvement of oscillation lemma

$$\operatorname*{osc}_{B_{R/2}} v \leq (1-\theta) \operatorname*{osc}_{B_R} v$$

and then iterate it to obtain

$$\operatorname*{osc}_{B_{2^{-k}}} v \leq (1-\theta)^k \operatorname*{osc}_{B_1} v$$

which imples a C^α modulus of continuity for $\alpha = \frac{\log(1-\theta)}{\log(\lambda)}$.



General strategy for proving Hölder continuity

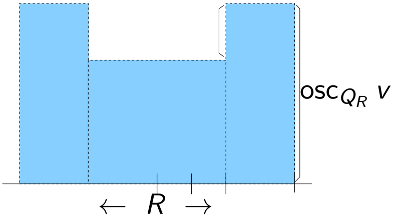
First prove an improvement of oscillation lemma

$$\operatorname*{osc}_{B_{R/2}} v \leq (1-\theta) \operatorname*{osc}_{B_R} v$$

and then iterate it to obtain

$$\operatorname*{osc}_{B_{2^{-k}}} v \leq (1-\theta)^k \operatorname*{osc}_{B_1} v$$

which imples a C^α modulus of continuity for $\alpha = \frac{\log(1-\theta)}{\log(\lambda)}$.



General strategy for proving Hölder continuity

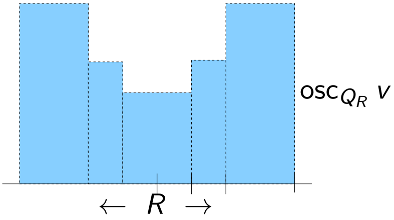
First prove an improvement of oscillation lemma

$$\operatorname*{osc}_{B_{R/2}} v \leq (1-\theta) \operatorname*{osc}_{B_R} v$$

and then iterate it to obtain

$$\operatorname*{osc}_{B_{2^{-k}}} v \leq (1-\theta)^k \operatorname*{osc}_{B_1} v$$

which imples a C^α modulus of continuity for $\alpha = \frac{\log(1-\theta)}{\log(\lambda)}$.



General strategy for proving Hölder continuity

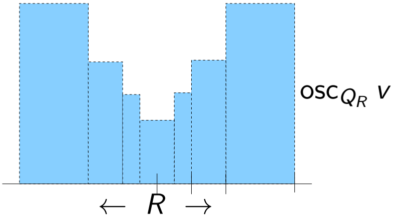
First prove an improvement of oscillation lemma

$$\operatorname*{osc}_{B_{R/2}} v \leq (1-\theta) \operatorname*{osc}_{B_R} v$$

and then iterate it to obtain

$$\operatorname*{osc}_{B_{2^{-k}}} v \leq (1-\theta)^k \operatorname*{osc}_{B_1} v$$

which imples a C^α modulus of continuity for $\alpha = \frac{\log(1-\theta)}{\log(\lambda)}$.



General strategy for proving Hölder continuity

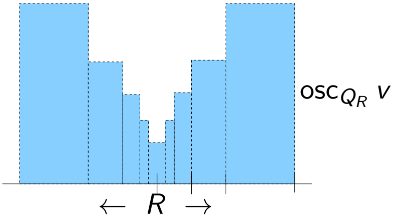
First prove an improvement of oscillation lemma

$$\operatorname*{osc}_{B_{R/2}} v \leq (1-\theta) \operatorname*{osc}_{B_R} v$$

and then iterate it to obtain

$$\operatorname*{osc}_{B_{2^{-k}}} v \leq (1-\theta)^k \operatorname*{osc}_{B_1} v$$

which imples a C^α modulus of continuity for $\alpha = \frac{\log(1-\theta)}{\log(\lambda)}$.



General strategy for proving Hölder continuity

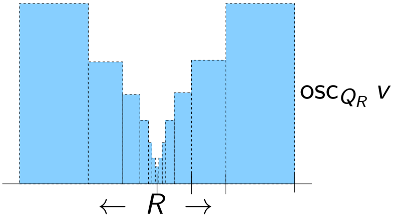
First prove an improvement of oscillation lemma

$$\operatorname*{osc}_{B_{R/2}} v \leq (1-\theta) \operatorname*{osc}_{B_R} v$$

and then iterate it to obtain

$$\operatorname*{osc}_{B_{2^{-k}}} v \leq (1-\theta)^k \operatorname*{osc}_{B_1} v$$

which imples a C^α modulus of continuity for $\alpha = \frac{\log(1-\theta)}{\log(\lambda)}$.



General strategy for proving Hölder continuity

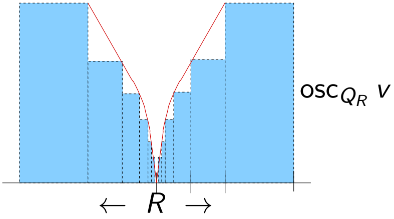
First prove an improvement of oscillation lemma

$$\operatorname*{osc}_{B_{R/2}} v \leq (1-\theta) \operatorname*{osc}_{B_R} v$$

and then iterate it to obtain

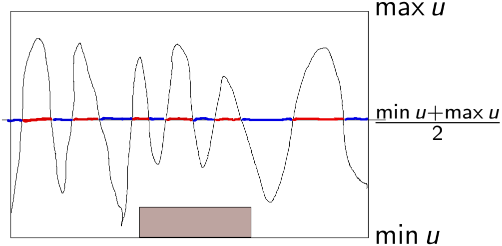
$$\operatorname*{osc}_{B_{2^{-k}}} v \leq (1-\theta)^k \operatorname*{osc}_{B_1} v$$

which imples a C^α modulus of continuity for $\alpha = \frac{\log(1-\theta)}{\log(\lambda)}$.

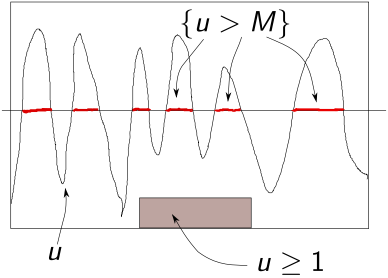


Which side improves?

The solution u will be either above or below its middle line in half of the points in B_1.



Improvement of oscillation from one side



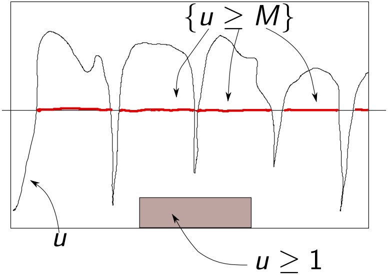
Lemma

Assume $u \geq 0$ **and** $a_{ij}(x)\partial_{ij} u \leq 0$ *in* B_1*. Moreover,*

$$|\{u \geq M\} \cap B_1| \geq \frac{1}{2}|B_1|,$$

Then $u \geq 1$ *in* $B_{1/2}$*.*

A weaker version of the lemma



Lemma

Assume $u \geq 0$ **and** $a_{ij}(x)\partial_{ij} u \leq 0$ *in* B_1*. Moreover,*

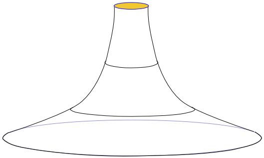
$$|\{u \geq M\} \cap B_1| \geq (1-\delta)|B_1|,$$

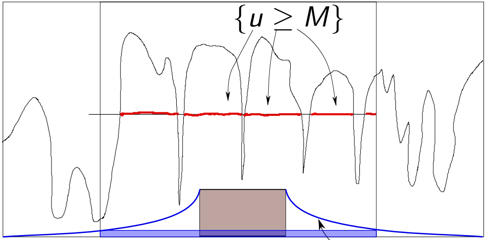
Then $u \geq 1$ *in* $B_{1/2}$.

This weaker version of the lemma + a barrier function + a covering argument $\Longrightarrow$ The previous (stronger) version of the lemma.

The barrier function

The barrier function $\varphi(x) = |x|^{-p}$ is a subsolution in $B_1 \setminus \{0\}$ for p large enough.





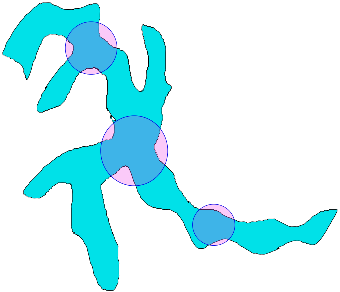
The barrier is used to expand the ball where we get a lower bound.

bounded below by the barrier

Growing ink spots and the L^ε estimate

The sets $A_k := \{u \geq M^k\} \cap B_{1/2}$ satisfy the following property.

For any ball B so that $|B \cap A_{k+1}| > (1-\delta)|B|$, then $B \subset A_k$.

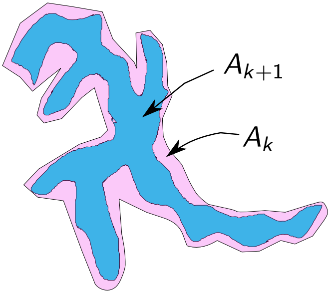


This property implies that $|A_{k+1}| \leq (1 - C\delta)|A_k|$ (covering argument). In particular $|A_k| < \frac{1}{2}|B_{1/2}|$ for k large.

Growing ink spots and the L^ε estimate

The sets $A_k := \{u \geq M^k\} \cap B_{1/2}$ satisfy the following property.

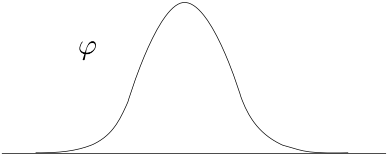
For any ball B so that $|B \cap A_{k+1}| > (1-\delta)|B|$, then $B \subset A_k$.



This property implies that $|A_{k+1}| \leq (1 - C\delta)|A_k|$ (covering argument). In particular $|A_k| < \frac{1}{2}|B_{1/2}|$ for k large.

The usual proof of the lemma (as in [CC])

Assume $u(x) \leq 1$ at some point $x \in B_{1/2}$.



Take $\varphi : B_1 \to \mathbb{R}$ to be a smooth function such that $\varphi \leq 0$ on ∂B_1 and $\varphi \geq 2$ in $B_{1/2}$.

Apply the ABP estimate to $u - \varphi$ and obtain

$$2 \leq \max(\varphi - u) \leq C \left(\int_{\{u-\varphi=\Gamma\}} |a_{ij}(x)\partial_{ij}\varphi(x)|^n dx \right)^{1/n} \leq C|\{u \leq \varphi\}|^{1/n}.$$

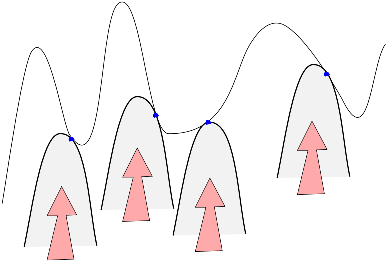
This proves the weaker lemma with $M = \max \varphi$.

Inconvenience: the ABP estimate is applied to $u - \varphi$. It is important the equation that $u - \varphi$ satisfies.

Variation: Sliding paraboloids from below.

This idea first appeared in the early work of X. Cabre (1997), and also played an important role in the work of O. Savin (2007).

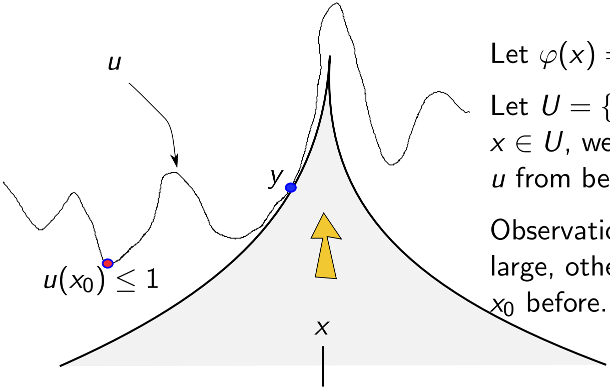
(also used in a recent paper by S. Armstrong and C. Smart)

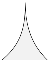


The measure of the set of contact points is bounded below!
The proof is similar to ABP, but it is more goemetrical, there is no subtraction and it provides more flexibility.

The inconvenience is now clearer: the touching point is typically by the top, where the gradient is small.

Our case: sliding cusps from below.



Let $\varphi(x) = -|x|^{1/2}$: 

Let $U = \{x : u(x) > M\}$. For any $x \in U$, we let $\varphi(\cdot - x) + q$ touch u from below at the point $y \in B_1$.

Observation: $u(y)$ cannot be large, otherwise it would touch at x_0 before. In particular $x \neq y$.

Let $m(y) := x$, and compute Dm, we will get $|Dm| \leq C$, which means that the measure of the set of contact points y is bounded below.

The computation

We have

$$\nabla u(y) = \nabla\varphi(y - x),$$
$$D^2 u(y) \geq D^2\varphi(y - x).$$

From the second inequality and the equation:
$|D^2 u(y)| \leq C|D^2\varphi(y - x)|$.
Writing $x = m(y)$ and taking derivatives in the first equality:
$D^2 u(y) = D^2\varphi(y - x)(I - Dm)$.
Thus

$$|Dm| = |D^2\varphi(y - x)^{-1}\left(D^2\varphi(y - x) - D^2 u(y)\right)| \leq C.$$

The last computation works magically because of the choice of φ so that all the eigenvalues of $D^2\varphi$ are comparable.

Some application

Theorem (Imbert, S.)

If u is a viscosity solution to a fully non linear equation of the form

$$|\nabla u|^{\gamma} F(D^2 u) = f,$$

with $\gamma > 0$, f bounded, and F uniformly elliptic, then $u \in C^{1,\alpha}$ for some $\alpha > 0$.

Something I cannot do

Conjecture

Let $u(x,t)$ be a function such that

$u_t = a_{ij}(x,t)\partial_{ij}u, \quad \textbf{wherever } |\nabla \mathbf{u}| \geq \gamma \text{ or } |\mathbf{u_t}| \geq \gamma$ in $B_1 \times (-1,0]$.

The coefficients a_{ij} are only assumed to satisfy

$$\lambda I \leq a_{ij}(x) \leq \Lambda I.$$

Then $u \in C^\alpha(B_{1/2} \times [-1/2, 0])$, with an estimate

$$\|u\|_{C^\alpha(B_{1/2}\times[-1/2,0])} \leq C\left(\lambda, \Lambda, n, \frac{\gamma}{\|u\|_{L^\infty}}\right)\|u\|_{L^\infty}.$$