On the continuity of the solution to drift-diffusion equations

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drift-diffusion equations

We look at functions u which solve the equation with drift and classical diffusion

$$u_t + b \cdot \nabla u - \triangle u = 0.$$

Where b is a vector field (depending on space and time).

fractional drift-diffusion equations

We look at functions u which solve the equation with drift and fractional diffusion

$$u_t + b \cdot \nabla u + (-\triangle)^s u = 0.$$

Where b is a vector field (depending on space and time). And $s \in (0,1)$.

The fractional Laplacian

$$\widehat{(-\triangle)^s u}(\xi) = |\xi|^{2s}\widehat{u}(\xi)$$

$$(-\triangle)^s u(x) = C_{n,s}\int_{\mathbb{R}^n}\frac{u(x)-u(x+y)}{|y|^{n+2s}}\,\mathrm{d}y$$

It is the infinitesimal generator of α-stable Levy processes.

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The question

For what kind of vector fields b, is the solution u going to remain smooth?

The characterization should be in terms of b having a bounded norm in some space like L^p, C^α, BMO, etc...

Can we get better regularity results when we assume that the drift b is **divergence free**?

Nonlinear equations

An a priori estimate for a linear drift-diffusion equation with minimal assumptions on the drift b can be applied to nonlinear equations, where b depends on the solution u.

Some scalar nonlinear equations with fractional diffusion.

Conservation Laws: $u_t + \operatorname{div} F(u) + (-\triangle)^s u = 0.$

Studied recently by Biler, Funaki, Woyczynski, Jourdain, Méléard, Droniou, Imbert, Czubak, Chan, Achleitner, Alibaud, Kiselev, Nazarov, Shterenberg, ...

Hamilton-Jacobi equation: $u_t + H(\nabla u) + (-\triangle)^s u = 0.$

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Proving regularity of the solution is relatively simple in the subcritical ($s > 1/2$) case, **interesting** in the critical case $s = 1/2$, and **false** in the supercritical case ($s < 1/2$).

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Some scalar nonlinear equations with fractional diffusion and divergence free drifts.

Surface quasi-geotrophic equation: $\theta_t + (R^{\perp}\theta)\cdot\nabla\theta + (-\triangle)^s\theta = 0.$

Studied recently by Constantin, Wu, Majda, Tabak, Fefferman, Caffarelli, Vasseur, Kiselev, Nazarov, Volberg, Hongjie Dong, Dabkowski, Cordoba, Cordoba, Vicol, Dong Li, and many more.

Flow in porous media: $\theta_t + u\cdot\nabla\theta + (-\triangle)^s\theta = 0$, where $u = (0, -\theta) - \nabla p$ is divergence free

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Other active scalar equations: $\theta_t + u\cdot\nabla\theta + (-\triangle)^s\theta = 0$, where $u = T\theta$ is divergence free.

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A priori estimate + linear estimates

Let us take a conservation law with fractional diffusion:

$$u_t + \operatorname{div} F(u) + (-\triangle)^s u = 0.$$

A priori estimate: $u \in L^\infty$, from the maximum principle.

Linearization: The function u satisfies the drift-diffusion equation

$$u_t + b \cdot \nabla u + (-\triangle)^s u = 0, \qquad \text{with } b_i = F_i'(u).$$

An estimate for drift-diffusion equations may give some extra regularity estimate for u.

Scaling of the equation

Suppose u solves

$$\partial_t u + b \cdot \nabla u + (-\triangle)^s u = 0.$$

Then $u_r(x,t) = u(rx, r^{2s}t)$ solves

$$\partial_t u_r + r^{2s-1} b(rx, r^{2s}t) \cdot \nabla u_r + (-\triangle)^s u_r = 0.$$

An assumption $b \in X$ is

- **Critical** if $\|r^{2s-1}b(rx, r^{2s}t)\|_X = \|b\|_X$ for any r.
- **Sub-critical** if $\|r^{2s-1}b(rx, r^{2s}t)\|_X = r^{\alpha}\|b\|_X$ for some $\alpha > 0$.
- **Super-critical** if $\|r^{2s-1}b(rx, r^{2s}t)\|_X = r^{-\alpha}\|b\|_X$ for some $\alpha > 0$.

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What to expect

In the **sub-critical** case, the diffusion is **stronger** than the drift in small scales. In the **super-critical** case, the diffusion is **weaker** than the drift in small scales. In the **critical** case, the diffusion and the drift are balanced at all scales.

In general we would expect the solution to a drift-diffusion equation to be

- **Differentiable** in the sub-critical case.
- **Hölder continuous** in the critical case.
- Possibly **discontinuous** in the super-critical case.

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Critical spaces for b

Depending on the power s of the Laplacian, the critical spaces for b vary.

- If $s < 1/2$, $b \in L^\infty([0,T], C^{1-2s})$ is critical.
- If $s = 1/2$, $b \in L^\infty([0,T], L^\infty)$ is critical.
- If $s > 1/2$, $b \in L^\infty([0,T], L^{n/(2s-1)})$ is critical.

Hölder regularity for elliptic drift-diffusion equations

Theorem (Stampacchia (1965) - Safonov (2010))

If u solves

$$b \cdot \nabla u - \triangle u = 0 \quad \textit{in } B_1$$

for an arbitrary vector field $b \in L^n$, then u is Hölder continuous in $B_{1/2}$ and there is an estimate

$$\|u\|_{C^\alpha(B_{1/2})} \leq C\|u\|_{L^2(B_1)}.$$

The constants C and α in Safonov's estimate depend on $\|b\|_{L^n}$ only. In Stampacchia's proof there is an implicit smallness condition.

Hölder regularity for parabolic drift-diffusion equations, with classical diffusion and no assumption on divergence.

Open problem

If u solves

$$u_t + b \cdot \nabla u - \triangle u = 0$$

for an arbitrary vector field $b \in L^\infty([0, T], L^n)$, will u become immediately Hölder continuous?

Hölder regularity for classical diffusion and divergence free drift

Theorem (Friedlander, Vicol (2011). Sverak, Seregin, S., Zlatos (2011))

If u solves

$$u_t + b \cdot \nabla u - \triangle u = 0$$

for an divergence-free vector field $b \in L^\infty([0,T], BMO^{-1})$, *then u becomes immediately Hölder continuous.*

Note that BMO^{-1} and L^n have the same scaling: if $u_r(x) = u(rx)$ then

$$\|u_r\|_{BMO^{-1}} = r^{-1}\|u\|_{BMO^{-1}} \qquad \|u_r\|_{L^n} = r^{-1}\|u\|_{L^n}.$$

The proof goes along the lines of DeGiorgi-Nash-Moser.

Hölder regularity for $s = 1/2$ and arbitrary divergence.

Theorem (S. 2010)

If u solves

$$u_t + b \cdot \nabla u + (-\triangle)^{1/2} u = 0$$

for an arbitrary bounded vector field b, then u becomes immediately Hölder continuous.

This theorem is the key to show that either the Hamilton-Jacobi equation or conservation laws have smooth solutions in the critical case $s = 1/2$.

The proof uses a new idea to obtain an improvement of oscillation lemma, which is then iterated to get the Hölder continuity. The proof uses strongly that the operator is non local.

Hölder regularity for divergence-free drifts and $s = 1/2$.

Theorem (Caffarelli and Vasseur. Annals of Math 2006)

If u solves

$$u_t + b \cdot \nabla u + (-\triangle)^{1/2} u = 0$$

for an arbitrary divergence free vector field b in $L^\infty(BMO)$, then u becomes immediately Hölder continuous.

The proof also follows the ideas from De Giorgi-Nash-Moser theorem.

This result implies the well posedness of the critical surface quasi-geostrophic equation. There is an independent proof given by Kiselev, Nazarov and Volberg, and another by Constantin and Vicol.

A result for $s \in (0, 1/2)$

Theorem (Constantin, Wu, AIHP 2009)

If u solves

$$u_t + b \cdot \nabla u + (-\triangle)^s u = 0$$

for some divergence-free vector field $b \in L^\infty(C^{1-2s})$, then u becomes immediately Hölder continuous.

Theorem (S. 2010)

If u solves

$$u_t + b \cdot \nabla u + (-\triangle)^s u = 0$$

for some vector field $b \in L^\infty(C^{1-2s})$, then u becomes immediately Hölder continuous.

Singularities with fractional diffusion

Theorem (S., Vicol, Zlatos 2012)

For any $s \in (0, 1/2)$ *and* $\alpha < 1 - 2s$, *there exists a* **divergence free** *vector field* $b \in C^\alpha(\mathbb{R}^2)$, **constant in time**, *so that a solution to*

$$u_t + b \cdot \nabla u + (-\triangle)^s u = 0$$

is smooth at time zero but becomes discontinuous at positive time.

Theorem (S., Vicol, Zlatos 2012)

For any $s \in [1/2, 1)$ *and* $p < 2/(2s - 1)$, *there exists a* **divergence free** *vector field* $b \in L^p(\mathbb{R}^2)$, **constant in time**, *so that a solution to*

$$u_t + b \cdot \nabla u + (-\triangle)^s u = 0$$

is smooth at time zero but becomes discontinuous at positive time.

The opposite claim

arXiv.org > math > arXiv:1007.3919

Search or Art

Mathematics > Analysis of PDEs

Remarks on a fractional diffusion transport equation with applications to the dissipative quasi-geostrophic equation

(Submitted on 22 Jul 2010 (v1), last revised 5 Dec 2011 (this version, v5))

In this article I study Hölder regularity for solutions of a transport equation based in the dissipative quasi-geostrophic equation. Following a recent idea of A. Kiselev and F. Nazarov, I will use the molecular characterization of local Hardy spaces in order to obtain information on Hölder regularity of such solutions. This will be done by following the evolution of molecules in a backward equation.

Comments: 29 pages
Subjects: **Analysis of PDEs (math.AP)**
Cite as: **arXiv:1007.3919v5 [math.AP]**

Submission history

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[v4] Tue, 12 Apr 2011 12:59:56 GMT (26kb)
[v5] Mon, 5 Dec 2011 11:01:29 GMT (27kb)

The author himself found a mistake in mid October, 2012.

Singularities with classical diffusion

Theorem (S., Vicol, Zlatos 2012)

For any $p < 2$, there exists a **divergence free** *vector field $b \in L^\infty(L^p(\mathbb{R}^2))$, so that a solution to*

$$u_t + b \cdot \nabla u - \triangle u = 0$$

is smooth at time zero but becomes discontinuous at positive time.

The same result would not hold for b constant in time

Continuity with classical diffusion

Theorem (S., Vicol, Zlatos 2012)

Let $b \in L^1(\mathbb{R}^2)$ *be a* **divergence free** *vector field* **independent of time**. *Let u be the solution to*

$$u_t + b \cdot \nabla u - \triangle u = 0$$

in $\mathbb{R}^2 \times (0, +\infty)$ *with smooth initial data. Then u remains continuous for all positive time.*

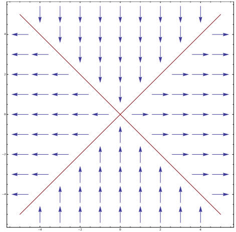
Notes:

- The result would not be true if we drop the zero divergence assumption.
- The result would not be true in higher dimensions.
- The result would not be true for fractional diffusion (as we saw before).

Construction of singularity for $b \in L^\infty$ and $s < 1/2$.

We will use the following vector field (independent of time)

$$b(x_1,x_2) = \begin{cases} (0,-1) & \text{if } x_2 > |x_1|, \\ (0,1) & \text{if } x_2 < -|x_1|, \\ (1,0) & \text{if } x_1 > |x_2|, \\ (-1,0) & \text{if } x_1 < -|x_2|. \end{cases}$$



The function u will satisfy the following symmetries (which are preserved by the equation):

$$u(x_1,x_2) = -u(x_1,-x_2),$$
$$u(x_1,x_2) = u(-x_1,x_2).$$

Comparison principle in the upper half plane

Let u be a solution and v a subsolution of the equation

$$\begin{aligned} u_t + b \cdot \nabla u + (-\triangle)^s u &= 0 \\ v_t + b \cdot \nabla v + (-\triangle)^s v &\leq 0 \end{aligned} \qquad \text{for } x_2 > 0.$$

Assume both u and v are odd in x_2, and that at time zero

$$u(x_1, x_2, 0) \geq v(x_1, x_2, 0) \text{ if } x_2 > 0.$$

Then for all positive time $t > 0$,

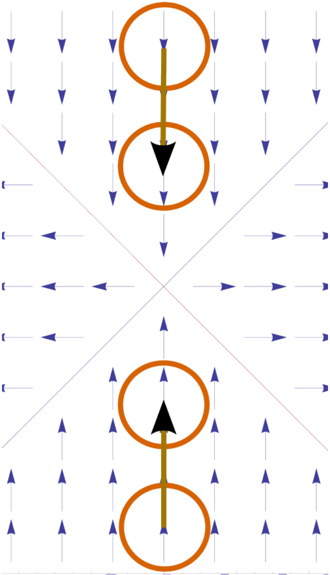
$$u(x_1, x_2, t) \geq v(x_1, x_2, t) \text{ if } x_2 > 0.$$

It is not a classical maximum principle because $(-\triangle)^s$ is nonlocal. The oddness of the functions plays a role in the proof.

The effect of transport

Let $u(x,0)$ be a smooth function, positive in $B_1(0,4)$, negative in $B_1(0,-4)$, and zero everywhere else.

If we ignore the effect of dissipation and concentrate on the drift, initially the function is just transported vertically towards $\{x_2 = 0\}$. After two seconds, the circles would move to $B_1(0,2)$ and $B_1(0,-2)$.



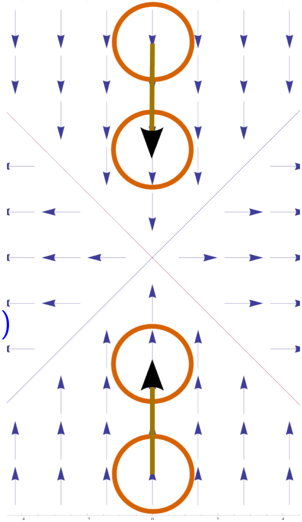
The effect of diffusion in the first two seconds

Let
$u(x,0) = \eta(x-(0,4)) - \eta(x-(0,-4))$
for some non negative smooth η
supported in B_1.

We can build a subsolution in the upper half plane for $t \in [0,2]$.
$v(x,t) =$
$e^{-Ct}(\eta(x-(0,4-t)) - \eta(x-(0,-4+t)))$

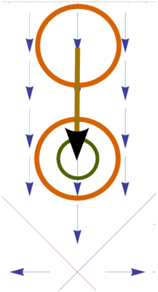
Thus $u \geq v$ for $t \in [0,2]$ and $x_2 > 0$.

The constant C is chosen to compensate the effect of dissipation.



After the two seconds

With divergence free drifts, we can never produce a self similar solution. Nonetheless, we can produce a self similar sub-solution.



We observe that for some constant C, we obtain

$$v(x,2) \geq e^{-C} v(2x,0) = e^{-C} \left(\eta(2x-(0,4)) - \eta(2x-(0,-4))\right)$$

So, we can rescale and start over after $t = 2$

Supercritical scaling

The idea is to iterate the process and find a sequence of constants C_k and a subsolution v such that

$$v(x, 4 - 2^{2-k}) \geq e^{-C_k} v(2^k x, 0)$$

But the equation is not invariant by scaling. In fact, $u_k = u(2^k x, 2^k t)$ satisfies the equation

$$\partial_t u_k + b \cdot \nabla u_k + 2^{(2s-1)k} (-\triangle)^s u_k = 0$$

For $s < 1/2$, the effect of the dissipation will decrease exponentially in small scales and we will obtain

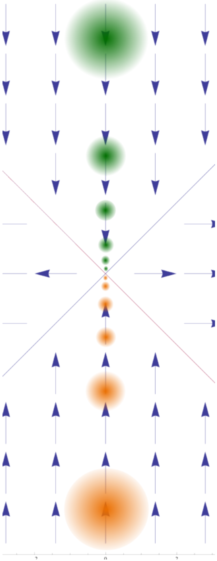
$$C_k = C \sum_{i=0}^{k-1} 2^{(2s-1)k}$$

which is bounded independently of k.

The shock

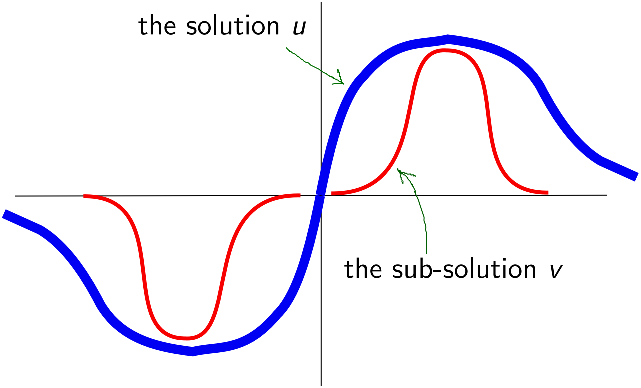
Effectively, we are finding a sequence of circles on the upper half plane $\{x_2 > 0\}$ where u is bounded below by a positive quantity, and their symmetric ones in $\{x_2 < 0\}$ where u is bounded above by a negative quantity.

At time $t = 4$ these circles meet at $x = 0$ and there is a discontinuity in u.



Another picture of the discontinuity

This is the graph of the values of u on the vertical axis $\{x_1 = 0\}$.



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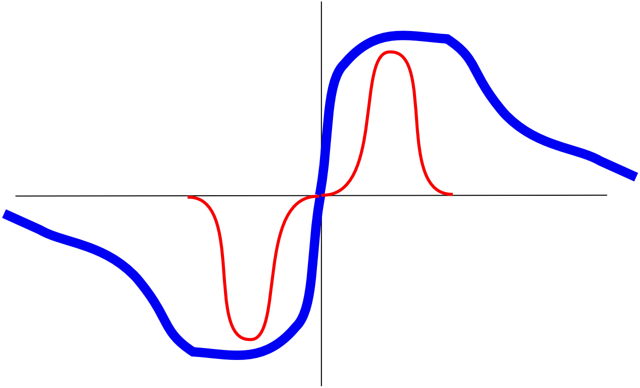

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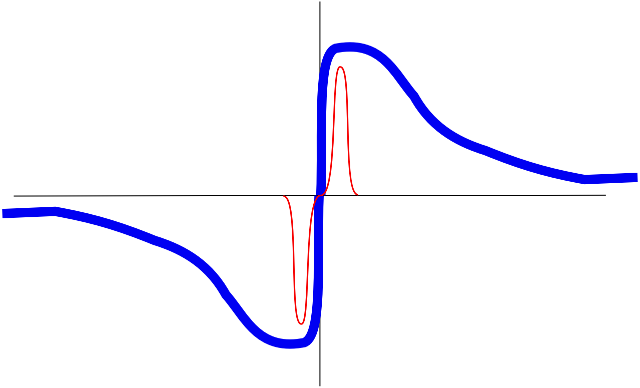


The case of continuity: how time independence helps

If u solves

$$u_t + b(x) \cdot \nabla u - \Delta u = 0$$

then u_t solves the same equation.

Since $\operatorname{div} b = 0$, we have the energy estimate both for u and u_t.

$$\left.\begin{array}{r} u \in L^2([0,T],H^1) \\ u_t \in L^2([0,T],H^1) \end{array}\right\} \Rightarrow u \in C([0,T],H^1)$$

Moreover, from the maximum principle, we can get a bound for u and u_t in L^∞.

Elliptic case

Fact

A function in H^1 which satisfies the maximum principle in every ball is continuous.

More precisely. If for every ball $B \subset B_1$, $\sup_B u = \sup_{\partial B} u$ and $\inf_B u = \inf_{\partial B} u$, then u has a modulus of continuity depending on its $\dot{H}^1$ norm only

$$\operatorname*{osc}_{B_r} u \leq \frac{C}{\sqrt{-\log r}} \|\nabla u\|_{L^2}.$$

This can be used to show that solutions to elliptic PDEs are continuous, as in the joint paper with Sverak, Seregin and Zlatos.

The idea can be traced back to **Lebesgue 1907**.

Proof of the fact

Let $A(r) = \operatorname{osc}_{\partial B_r} u = \operatorname{osc}_{B_r}$, which is increasing.
We have

$$C = \int_{B_1} |\nabla u|^2 \geq \int_{\varepsilon}^{1} \int_{\partial B_1} \frac{|\partial_\theta u(r\theta)|^2}{r} + |\partial_r u(r\theta)|^2 \, \mathrm{d}\theta \, \mathrm{d}r$$

$$\geq \int_{\varepsilon}^{1} \frac{c}{r} \left(\int_{\partial B_1} |\partial_\theta u(r\theta)| \, \mathrm{d}\theta \right)^2 \mathrm{d}r \geq c \int_{\varepsilon}^{1} \frac{A(r)^2}{r} \, \mathrm{d}r$$

$$\geq c(-\log \varepsilon) A(\varepsilon)^2.$$

Parabolic difficulty

The estimate $u \in C([0,T], H^1)$ plus the **parabolic maximum principle** is **not** enough to obtain a modulus of continuity.

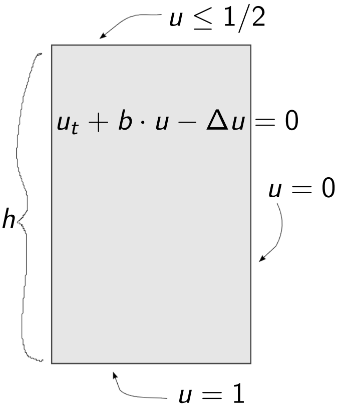
In fact, for any function $f \in H^1(\mathbb{R}^2) \setminus C(\mathbb{R}^2)$, $u(x,t) = f(x)$ is a counterexample.

We need to use the equation further.

Another estimate independent of the drift

Lemma

There exists an $h > 0$, **independent of** *b, so that the solution to the equation in $[0, h] \times B_1$ with $u(0, x) = 1$ and $u(t, x) = 0$ for all $x \in \partial B_1$, satisfies $u \leq 1/2$ on $\{h\} \times B_1$.*



The proof is based on a pointwise estimate on the fundamental solution originally due to John Nash.

Almost elliptic maximum principle

Lemma

On each time slice t, the function u solving

$$u_t + b \cdot \nabla u - \triangle u = 0$$

satisfies an approximate maximum principle

$$\sup_{B_r} u \leq \sup_{\partial B_r} u + C \|u_t\|_{L^\infty} r^2,$$

for a constant C independent of b.

The main ingredient of the proof is the previous Lemma, which says that in the parabolic maximum principle the bottom cannot dominate the value of the maximum in parabolic cylinders with certain proportions.