Results in nonlocal elliptic equations

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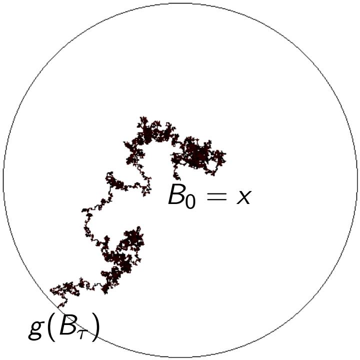
joint work with Luis Caffarelli

Outline

Brownian motion $\longrightarrow$ Laplace Equation



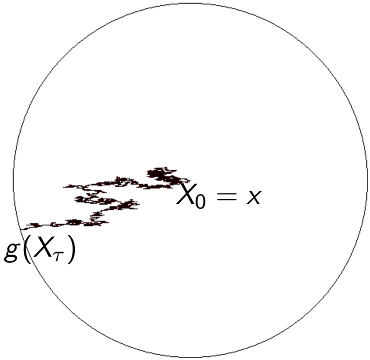
Let $g : \partial\Omega \to \mathbb{R}$.
Let B be a Brownian motion.
$B_0 = x$

B hits $\partial\Omega$ at B_τ.

Let $u(x) = \mathbb{E}(g(B_\tau))$
Then

$$\triangle u = 0 \text{ in } \Omega$$
$$u = g \text{ on } \partial\Omega$$

Diffusions → Elliptic PDE with coefficients



Let $g : \partial\Omega \to \mathbb{R}$.
$X = \sqrt{A} dB$ $(A \geq 0)$
$X_0 = x$

X hits $\partial\Omega$ at X_τ.

Let $u(x) = \mathbb{E}(g(X_\tau))$
Then

$$A_{ij}\partial_{ij} u = 0 \text{ in } \Omega$$
$$u = g \text{ on } \partial\Omega$$

Stochastic control → Hamilton-Jacobi-Bellman equation

Suppose we can choose the value of the coefficients a_{ij} at every point from a family of choices a_{ij}^{α} (α is our control). We want to minimize the expected value of $g(X_\tau)$.
The function

$$u(x) = \inf_{\text{all choices of } \alpha \text{ at every point}} \mathbb{E}(g(X_\tau))$$

solves the equation

$$\inf_{\alpha} a_{ij}^{\alpha} \partial_{ij} u = 0 \text{ in } \Omega$$

(generic $F(D^2u)$ for F concave)

Evans and Krylov 1982: Solutions of these equations are $C^{2,\alpha}$.

Optimal stopping → obstacle problem

$$u(x) = \sup_{\tau} E(\varphi(X_\tau^x))$$

where X_t^x is a Brownian motion starting at x and τ is any stopping time.

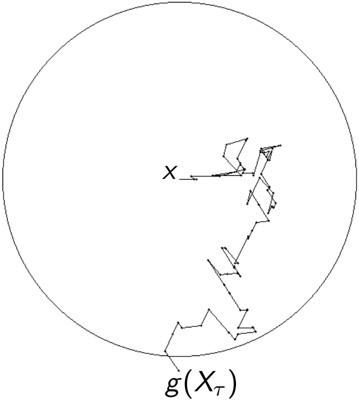
$$\begin{aligned} \triangle u &= 0 \quad \text{where } u > \varphi \text{ (at the points of no stop)} \\ \triangle u &\leq 0 \quad \text{everywhere in the domain} \\ u &\geq \varphi \end{aligned}$$

A similar model is used in financial mathematics for pricing American options

Frehse 1972: Solutions of this equation are $C^{1,1}$.
Caffarelli 1977: The free boundary is smooth except at a small set of singular points.

Processes with jumps

Integro-differential equations (instead of PDEs) arise when we consider processes X_t with discontinuities. Processes that jump from one point to another.



X_t is a purely jump Lévy process.
$u(x) = \mathbb{E}u(X_\tau)$ solves

$$\int_{\mathbb{R}^n} (u(x+y) - u(x))K(y)\,\mathrm{d}y = 0 \text{ in } \Omega$$

$$u = g \text{ outside } \Omega$$

General kernels

$$\int_{\mathbb{R}^n} (u(x+y) - u(x))K(y)\,\mathrm{d}y = 0$$

The kernel $K(y)$ represents the frequency of jumps in every direction y.

- $K(y) \geq 0$ for every y.
- $\int_{\mathbb{R}^n} \min(y^2, 1)K(y)\,\mathrm{d}y < +\infty$.

(We will only consider symmetric kernels $K(y) = K(-y)$)

Fractional Laplacian

The most typical case is

$$Lu(x) = -(-\triangle)^{\sigma/2} u(x) = c \int_{\mathbb{R}^n} (u(x+y) - u(x)) \frac{1}{|y|^{n+\sigma}} \, \mathrm{d}y$$

which is the fractional Laplacian

$$-\widehat{(-\triangle)^{\sigma}}^{/2} u(\xi) = -|\xi|^{\sigma} \widehat{u}(\xi).$$

$-(-\triangle)^{\sigma/2}$ is for integro-differential equations what $\triangle$ is for elliptic PDEs.

The constant c depends on n and σ, and $c \approx (2-\sigma)$ as $\sigma \to 2$.

Obstacle problem for the fractional laplacian

$$u(x) = \sup_{\tau} E(\varphi(X^x_\tau))$$

where X^x_t is an α-stable Lévy process starting at x and τ is any stopping time.

$$\begin{aligned}
(-\triangle)^{\alpha/2} u &= 0 \quad \text{where } u > \varphi \text{ (at the points of no stop)} \\
(-\triangle)^{\alpha/2} u &\leq 0 \quad \text{everywhere in } \mathbb{R}^n \\
u &\geq \varphi
\end{aligned}$$

A similar model is used in financial mathematics for pricing American options

Caffarelli, Salsa, S. 2008: Solutions of this equation are $C^{1,\alpha/2}$. and the free boundary is smooth except at some singular points

elliptic integro-differential equations

We say

$$\int_{\mathbb{R}^n} (u(x+y) - u(x)) K(y) \,\mathrm{d}y = 0$$

is uniformly elliptic of order $\sigma \in (0,2)$ if

- $K(y) = K(-y)$ for every y.
- $(2-\sigma)\frac{\lambda}{|y|^{n+\sigma}} \leq K(y) \leq (2-\sigma)\frac{\Lambda}{|y|^{n+\sigma}}$

Nonlinear integro-differential equations

In the same way as for diffusions, we can consider stochastic control problems with jumps to obtain nonlinear equations of the form

$$0 = \mathrm{I}u(x) := \sup_{\alpha} \int_{\mathbb{R}^n} (u(x+y) - u(x)) K_\alpha(y) \, \mathrm{d}y$$

Recovering second order PDEs

Note that the classical PDEs can be recovered from integro-differential equations in several ways. For example:

$$\begin{aligned} \triangle u(x) &= \lim_{s\to 1} -(-\triangle)^s u(x) \\ &= \lim_{r\to 0} \frac{c}{r^{n+2}} \int_{B_r} u(x+y) - u(x)\, \mathrm{d}y \end{aligned}$$

Dirichlet Problem

The natural Dirichlet problem is

$$\begin{aligned} \mathrm{I}u(x) &= 0 \text{ in } \Omega \\ u(x) &= g(x) \text{ in } \mathbb{R}^n \setminus \Omega \end{aligned}$$

Note that the boundary condition is given in the whole complement of the domain: $\mathbb{R}^n \setminus \Omega$. This is because of the nonlocal character of the equation.

It can be shown in fairly good generality that this problem admits a unique *viscosity* solution.

What about the regularity?

Uniformly elliptic PDEs

Given $0 < \lambda < \Lambda$, for second order elliptic PDEs $F(D^2 u) = 0$, ellipticity is defined by the following property of the function F

$$\lambda \|Y\| \leq F(X+Y) - F(X) \leq \Lambda \|Y\|$$

every time Y is a positive matrix.

If F is smooth, this is equivalent to the matrix inequality

$$\lambda I \leq \frac{\partial F}{X_{ij}} \leq \Lambda I$$

Pucci's maximal operator

Uniform ellipticity can also be described by means of the extremal Pucci operators:

$$M^+(X) = \lambda(\text{sum of negative eigenvalues of } X) + \Lambda(\text{sum of positive eigenvalues of } X)$$

$$M^-(X) = \Lambda(\text{sum of negative eigenvalues of } X) + \lambda(\text{sum of positive eigenvalues of } X)$$

Now, $F(D^2u) = 0$ is a uniformly elliptic equation if

$$M^-(Y) \leq F(X+Y) - F(X) \leq M^+(Y)$$

Regularity results for fully nonlinear PDEs

- Krylov-Safonov Harnack inequality (1979) $\Rightarrow$ Hölder estimates.

  If u is a bounded function in B_1 such that $M^+u \geq 0$ and $M^-u \leq 0$ in B_1, then u is Hölder continuous in $B_{1/2}$.

- $C^{1,\alpha}$ regularity.

  If u is a solution to a uniformly elliptic fully nonlinear equation $F(D^2u) = 0$ in B_1 then $u \in C^{1,\alpha}$ in $B_{1/2}$ for some $\alpha > 0$.

- Evans-Krylov theorem (1982)

  If u is a solution to a **convex** *uniformly elliptic fully nonlinear equation $F(D^2u) = 0$ in B_1 then $u \in C^{2,\alpha}$ in $B_{1/2}$ for some $\alpha > 0$.*

Nonlocal extremal operators

The Pucci extremal operators are also given by the formula

$$M^+(D^2u) = \sup_{\lambda I \leq \{a_{ij}\} \leq \Lambda I} a_{ij}\partial_{ij}u$$

$$M^-(D^2u) = \inf_{\lambda I \leq \{a_{ij}\} \leq \Lambda I} a_{ij}\partial_{ij}u$$

An integro-differential analog of order σ would be

$$M_\sigma^+ u(x) = \sup_{\substack{\lambda \leq a(y) \leq \Lambda \\ a(y)=a(-y)}} (2-\sigma)\int (u(x+y)-u(x))\frac{a(y)}{|y|^{n+\sigma}}\,\mathrm{d}y$$

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An integro-differential analog of order σ would be

$$M^+_\sigma u(x) = \sup_{\lambda \le a(y) \le \Lambda} \frac{2-\sigma}{2} \int (u(x+y) + u(x-y) - 2u(x)) \frac{a(y)}{|y|^{n+\sigma}} \, \mathrm{d}y$$

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An integro-differential analog of order σ would be

$$M_\sigma^+ u(x) = \frac{2-\sigma}{2} \int_{\mathbb{R}^n} \frac{\Lambda(u(x+y)+u(x-y)-2u(x))^+ - \lambda(\dots)^-}{|y|^{n+\sigma}} \, \mathrm{d}y$$

Uniform ellipticity for nonlocal equations

We say that a nonlocal operator I is uniformly elliptic of order σ if

$$M_\sigma^- v(x) \leq \mathrm{I}(u+v)(x) - \mathrm{I}u(x) \leq M_\sigma^+ v(x)$$

(σ is always in $(0,2)$)

Examples:

$$Lu(x) = \int (u(x+y) - u(x)) \frac{a(y)}{|y|^{n+\sigma}} \, \mathrm{d}y \qquad \text{for } \lambda \leq a \leq \Lambda \text{ and } a(y) = a(-y)$$

$$\mathrm{I}u(x) = \inf_\alpha \sup_\beta L_{\alpha\beta} u(x) \qquad \text{for } L_{\alpha\beta} \text{ linear as the one above}$$

$$\mathrm{I}u(x) = \int \frac{G(u(x+y) + u(x-y) - 2u(x))}{|y|^{n+\sigma}} \, \mathrm{d}y \qquad G \text{ monotone Lipschitz and } G(0) = 0$$

The Harnack inequality

Theorem (Caffarelli, S.)

Let $u \geq 0$ in $\mathbb{R}^n$, $M_\sigma^- u \leq 0$ and $M_\sigma^+ u \geq 0$ in B_2.
Then

$$\sup_{B_1} u \leq C \inf_{B_1} u$$

Important: The constant C does not blow up as $\sigma \to 2$.

We can understand the condition $M_\sigma^- u \leq 0$ and $M_\sigma^+ u \geq 0$ as that there is some kernel $a(x, y)$ such that

$$\int (u(x+y) + u(x-y) - 2u(x)) \frac{(2-\sigma)a(x,y)}{|y|^{n+\sigma}} \,\mathrm{d}y = 0$$

with $\lambda \leq a(x, y) \leq \Lambda$, and a can be very discontinuous.

Hölder estimates

Theorem (Caffarelli, S.)

Let $u \in L^\infty(\mathbb{R}^n)$, $M_\sigma^- u \leq 0$ and $M_\sigma^+ u \geq 0$ in B_2.
Then $u \in C^\alpha(B_1)$ and

$$u_{C^\alpha(B_1)} \leq C \sup_{\mathbb{R}^n} |u|$$

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with $\lambda \leq a(x, y) \leq \Lambda$, and a can be very discontinuous.

Differentiability of solutions

Theorem (Caffarelli, S.)

If I *is a nonlocal elliptic operator of order* σ *and* u *is a bounded function such that* $\mathrm{I}u = 0$ *in* B_1*, then* $u \in C^{1+\alpha}(B_{1/2})$ *and*

$$u_{C^{1+\alpha}(B_{1/2})} \leq C \left(\sup_{\mathbb{R}^n} |u| + |I0| \right)$$

Important: The constant C does not blow up as $\sigma \to 2$.

More regular solutions for concave problems

Theorem (Caffarelli, S.)

If I *is a* **concave** *nonlocal elliptic operator of order* σ *and* u *is a bounded function such that* $\mathrm{I}u = 0$ *in* B_1*, then* $u \in C^{\sigma+\alpha}(B_{1/2})$ *and*

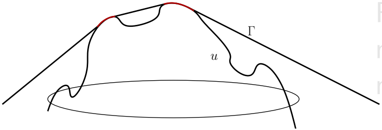
$$u_{C^{\sigma+\alpha}(B_{1/2})} \leq C \left(\sup_{\mathbb{R}^n} |u| + |I0| \right)$$

Important: The constant C does not blow up as $\sigma \to 2$.
α can also be chosen independently of σ.

Alexandroff-Bakelman-Pucci estimate

The proof of Harnack inequality for elliptic PDEs of second order is based on the ABP estimate: if $M^+ u \geq -f$ in B_1, $u \leq 0$ on ∂B_1, and Γ is the concave envelope of u in B_2 then

$$c(\max_{B_1} u)^n \leq |\nabla\Gamma(B_1)| = \int_{\{u=\Gamma\}} \det(-D^2\Gamma)\,\mathrm{d}x \leq C \int_{\{u=\Gamma\}} f^n\,\mathrm{d}x$$



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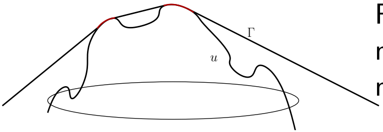
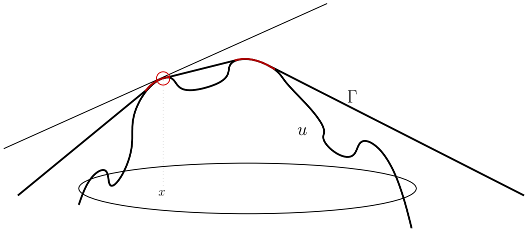


For integro differential equations, we need some alternative way to measure $\{u = \Gamma\}$

No cancellations in the integral

Let $x \in \{u = \Gamma\}$, $-f(x) \leq M^+ u(x)$

$$M^+_\sigma u(x) = \int \frac{\Lambda(u(x+y) + u(x-y) - 2u(x))^+ - \lambda(u(x+y) + u(x-y) - 2u(x))^-}{|y|^{n+\sigma}} \, \mathrm{d}y$$



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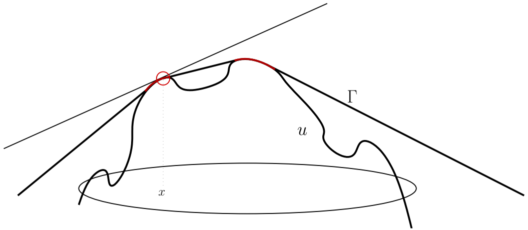
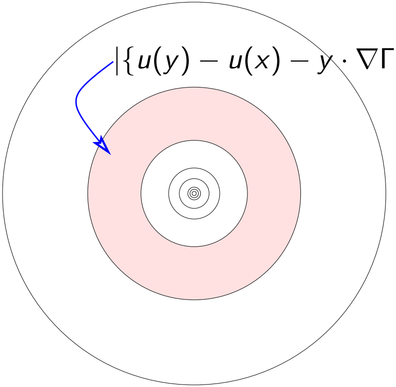


One good ring

We compare $u(y) - u(x) - y \cdot \nabla\Gamma(x)$ with $A|y|^2$.

$$\int (2-\sigma) \frac{\lambda(u(x+y)+u(x-y)-2u(x))^-}{|y|^{n+\sigma}} \, \mathrm{d}y \le c \; \frac{f(x)}{A} \int_{B_r} \frac{(2-\sigma)A|y|^2}{|y|^{n+\sigma}} \, \mathrm{d}y$$

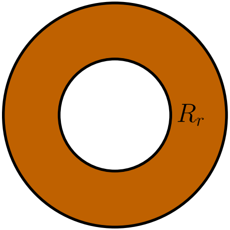
$$|\{u(y) - u(x) - y \cdot \nabla\Gamma(x) \le A|y|^2\}| \le C \frac{f(x)}{A} r^n$$



Catching up with the integrals

Lemma: Assume $M_\sigma^+ u \geq -f$ in B_1 (where M_σ^+ is now the maximal operator of order σ). $u \leq 0$ in $\mathbb{R}^n \setminus B_1$ and Γ is the concave envelope of u in B_3. If $u(x) = \Gamma(x)$, for every $A > 0$ there is ring $R_r(x)$ such that

$$|R_r \cap \{u(y) \leq u(x) - (y-x) \cdot \nabla\Gamma(x) - Ar^2\}| \leq C \frac{f(x)}{A} r^n$$



$\Gamma(y) \leq u(x) - (y-x) \cdot \nabla\Gamma(x) - Ar^2$ for all y in $B_{r/2}$

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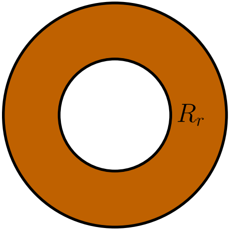

$\Gamma(y) \leq u(x) - (y-x) \cdot \nabla\Gamma(x) - Ar^2$ for all y in $B_{r/2}$

Consequences of the lemma

Around each point $x \in \{u = \Gamma\}$ there is a (small) ball $B_r(x)$ such that

- $u \geq \Gamma - Cf(x)r^2$ in a large proportion of $B_r(x)$.
- $|\nabla\Gamma(B_r(x))| \leq Cf(x)^n |B_r|$.

By covering the whole contact set $\{u = \Gamma\}$ with a subfamily of such balls with finite overlapping we find

$$|\nabla\Gamma(B_1)| \leq C \left|\left\{u(x) \geq \Gamma(x) - Cr_0^2\right\}\right|$$

(r_0 is the maximum possible value of r, which depends on σ)

nonlocal ABP

Thus we obtain

$$c(\max u)^n \leq |\nabla\Gamma(B_1)| \leq C\left|\left\{u(x) \geq \Gamma(x) - Cr_0^2\right\}\right|$$

which is good enough to carry out the rest of the proof of Harnack inequality and Hölder estimates.

Difference of solutions

If u and v are solutions to the same equation $\mathrm{I}u = \mathrm{I}v = 0$, then their difference solves

$$M^-(u-v) \le 0 \le M^+(u-v)$$

One can understand this as a linear equation with a priori discontinuous coefficients.

$$\int_{\mathbb{R}^n} ((u-v)(x+y) + (u-v)(x-y) - 2(u-v)(x)) \, K(x,y) \, \mathrm{d}y = 0$$

where $(2-\sigma)\frac{\lambda}{|y|^{n+\sigma}} \le K(x,y) \le (2-\sigma)\frac{\Lambda}{|y|^{n+\sigma}}$ with no continuity a priori in x.

More on difference of solutions

If u and v are solutions to the same equation $\mathrm{I}u = \mathrm{I}v = 0$, then

$$M^{-}(u-v) \leq 0 \leq M^{+}(u-v)$$

which also implies that the integrals of positive and negative incremental quotients

$$\int_{\mathbb{R}^n} ((u-v)(x+y) + (u-v)(x-y) - 2(u-v)(x))^{\pm} \frac{1}{|y|^{n+\sigma}} \, \mathrm{d}y$$

are comparable.

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$$\int_{\mathbb{R}^n} \delta_y(u-v)(x)^{\pm} \frac{1}{|y|^{n+\sigma}} \,\mathrm{d}y$$

are comparable.

$C^{1,\alpha}$ estimates

The differential quotient $w_h = \frac{u(x+he)-u(x)}{h}$ satisfies an equation

$$M^- w_h \leq 0 \leq M^+ w_h$$

$\implies w_h$ is C^α independently of h, and $u \in C^{1,\alpha}$.

(there is a technical difficulty because u may not be C^1 outside of the domain)

Concavity

If I is concave and u is a solution of $\mathrm{I}u = 0$ then a mollification is a subsolution.

$$\mathrm{I}(u * \eta) \geq 0$$

In particular for

$$\delta_y u(x) := (u(x+y) + u(x-y) - 2u(x))$$

we have

$$M^+ \delta_y u(x) \geq 0$$

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In particular for

$$\int_{B_h} \delta_y u(x) K(y)\, \mathrm{d}y \cong u * K - \left(\int K \, \mathrm{d}y \right) u$$

we have

$$M^+ \int_{B_h} \delta_y u(x) K(y)\, \mathrm{d}y \geq 0$$

Evans-Krylov theorem

For the proof of Evans-Krylov theorem, it is not enough to have

$$M^+ \delta_y u(x) \geq 0$$

to get $u \in C^{2,\alpha}$.

The equation has to be used further. In particular that

$$\|D^2 u^+\| \approx \|D^2 u^-\|$$

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The equation has to be used further. In particular that

$$\|(D^2u(x) - D^2u(y))^+\| \approx \|(D^2u(x) - D^2u(y))^-\|$$

What is used

Positive and negative parts of the integral control each other.

$$\int (\delta_y u(x) - \delta_y u(0))^{+} \frac{(2-\sigma)}{|y|^{n+\sigma}} \, \mathrm{d}y \approx \int (\delta_y u(x) - \delta_y u(0))^{-} \frac{(2-\sigma)}{|y|^{n+\sigma}} \, \mathrm{d}y$$

Linear integral operators are subsolutions

$$M^{+} \int (\delta_y u(x) - \delta_y u(0)) K(y) \, \mathrm{d}y \geq 0$$

for any $K \geq 0$.

Steps in the proof

Step 1. Prove that the integrals converge absolutely:

$$\int |\delta_y u(x)| \frac{(2-\sigma)}{|y|^{n+\sigma}} \, \mathrm{d}y \leq C$$

Step 2. Prove that the function is $C^{\sigma+\alpha}$.

Scheme of step 2.

We prove that

$$P(x) := \int (\delta_y u(x) - \delta_y u(0))^{+} \frac{(2-\sigma)}{|y|^{n+\sigma}} \, \mathrm{d}y \leq C|x|^{\alpha}$$

This implies that

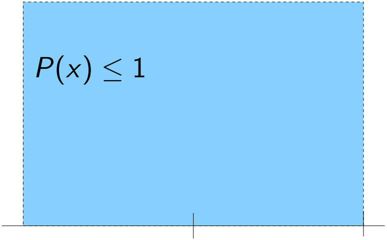
$$\int |\delta_y u(x) - \delta_y u(0)| \frac{(2-\sigma)}{|y|^{n+\sigma}} \, \mathrm{d}y \leq C|x|^{\alpha}$$

which immediately implies that $u \in C^{\sigma+\alpha}$.

Inductive argument

We show that for every $r \in (0, 1)$,

$$\sup_{B_{r/2}} P(x) \leq (1 - \theta) \sup_{B_r} P(x) \qquad \text{for some } \theta > 0$$



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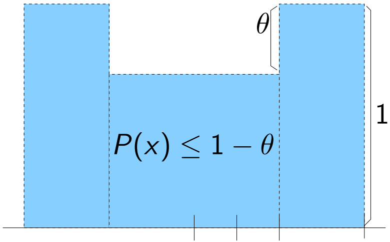

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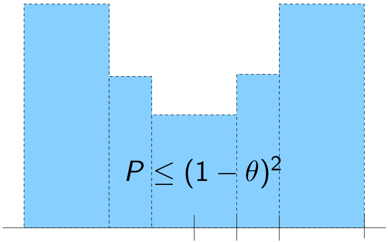

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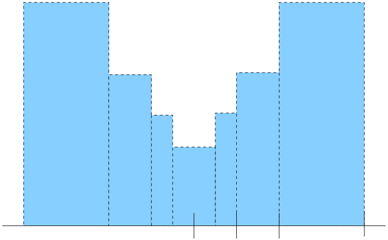

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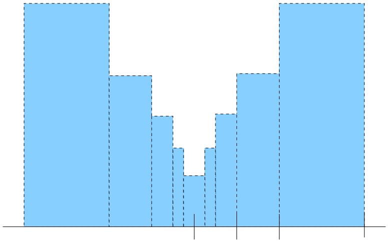

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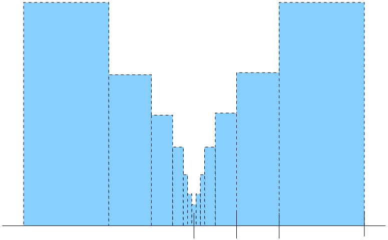

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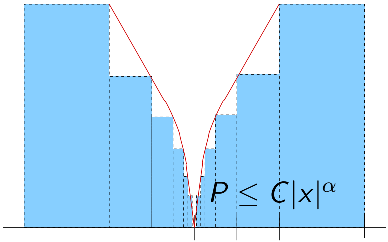

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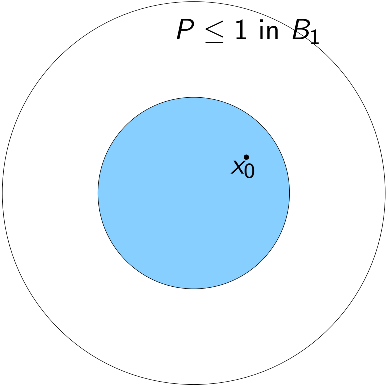
The inductive step



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$$P(x_0) = \int (\delta_y u(x_0) - \delta_y u(0))^{+} \frac{(2-\sigma)}{|y|^{n+\sigma}} \,\mathrm{d}y$$

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Recall

$$P(x_0) = \int (\delta_y u(x_0) - \delta_y u(0)) \frac{(2-\sigma)}{|y|^{n+\sigma}} \chi_A \, \mathrm{d}y$$

where $A = \{y : \delta_y u(x) - \delta_y u(0) > 0\}$

A tool: weak Harnack inequality

The following versions of the *weak Harnack inequality* are available for sub and super-solutions.

Theorem

Let $u \geq 0$ *in* $\mathbb{R}^n$ *and* $M^- u \leq 0$ *in* B_1 *(supersolution).*

$$|\{u > t\} \cap B_1| \leq C t^{-\varepsilon} \inf_{B_{1/2}} u \qquad \text{for every } t > 0.$$

Theorem

If $M^+ u \geq 0$ *in* B_1 *(subsolution) then*

$$u(x) \leq C \int_{\mathbb{R}^n} \frac{|u(y)|}{1 + |y|^{n+\sigma}} \, \mathrm{d}y \qquad \text{in } B_{1/2}$$

First posibility

Since $M^+\left(\int(\delta_y u(x)-\delta_y u(0))\frac{(2-\sigma)}{|y|^{n+\sigma}}\chi_A\,\mathrm{d}y\right)\geq 0$, if we had

$$\int(\delta_y u(x)-\delta_y u(0))\frac{(2-\sigma)}{|y|^{n+\sigma}}\chi_A\,\mathrm{d}y\leq(1-C\theta)$$

in a fraction of B_1, we would obtain

$$P(x_0)=\int(\delta_y u(x_0)-\delta_y u(0))\frac{(2-\sigma)}{|y|^{n+\sigma}}\chi_A\,\mathrm{d}y\leq 1-\theta$$

by weak Harnack inequality.
But what if the opposite inequality holds in most of B_1?

Second posibility

If

$$\int (\delta_y u(x) - \delta_y u(0)) \frac{(2-\sigma)}{|y|^{n+\sigma}} \chi_A \,\mathrm{d}y \geq (1 - C\theta)$$

in most of B_1, that means that the same choice of set A is approximately optimal to compute $P(x)$ in most of B_1.

$$\int (\delta_y u(x) - \delta_y u(0))^+ \frac{(2-\sigma)}{|y|^{n+\sigma}} \,\mathrm{d}y \approx \int (\delta_y u(x) - \delta_y u(0)) \frac{(2-\sigma)}{|y|^{n+\sigma}} \chi_A \,\mathrm{d}y$$

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with the same set A for most $x \in B_1$.

The punchline

If we have $\int (\delta_y u(x) - \delta_y u(0))^{+} \frac{(2-\sigma)}{|y|^{n+\sigma}} \, \mathrm{d}y$ very positive in most of B_1

But then we can apply weak Harnack and obtain that $\int (\delta_y u(x) - \delta_y u(0)) \frac{(2-\sigma)}{|y|^{n+\sigma}} \chi_{A^c} \, \mathrm{d}y$ is strictly negative for all $x \in B_{1/2}$

This is a contradiction at $x = 0$!

This finishes the proof of the inductive step $\Rightarrow P(x) \leq C|x|^{\alpha} \Rightarrow u \in C^{\sigma+\alpha}$.

The punchline

If we have $\int (\delta_y u(x) - \delta_y u(0))^{-} \frac{(2-\sigma)}{|y|^{n+\sigma}} \, \mathrm{d}y$ very negative in most of B_1

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