Lower dimensional obstacle problems and the fractional Laplacian

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joint work with Luis Caffarelli and Sandro Salsa

Classical Obstacle problem

Let us consider a surface given by the graph of a function u.

u is a function solving $\triangle u = 0$ for fixed boundary data. (we can think of an elastic membrane)

Let us now slide an obstacle from below. The surface must stay above it. For a given obstacle φ, we obtain a function $u \geq \varphi$, that will try to be as harmonic as possible.

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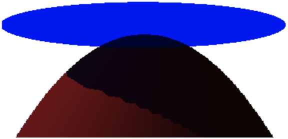

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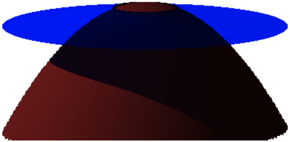

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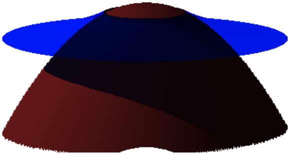

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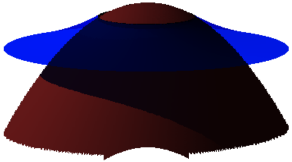

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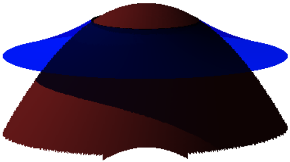

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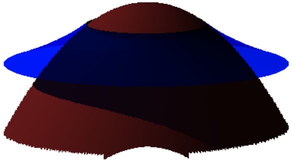

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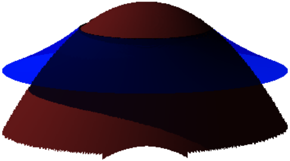

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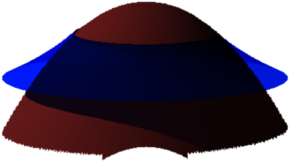

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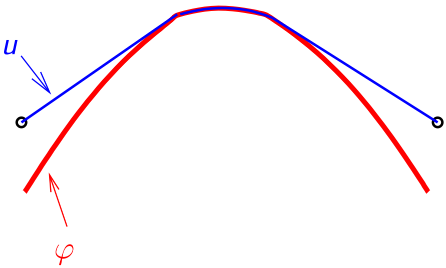
$\triangle u = 0$ where $u > \varphi$, since there u is free to move

$\triangle u \leq 0$ everywhere, since the surface pushes down

$u \geq \varphi$



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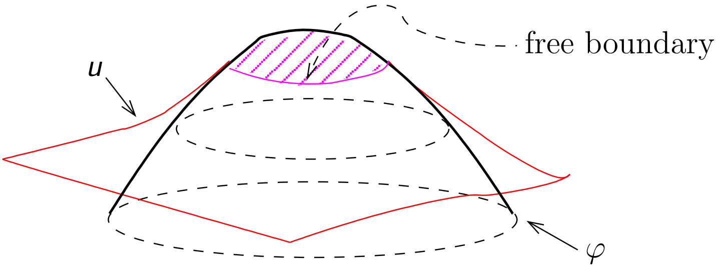

Regularity results

The regularity results for the classical obstacle problems are

- The function $u \in C^{1,1}$ (Frehse 1972).
- The free boundary is smooth besides a small singular set (Caffarelli 1977).

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Stochastic approach

$$u(x) = \sup_{\tau} E(\varphi(B^x_\tau))$$

where B^x_t is Brownian motion starting at x and τ is any stopping time.

Then

$$\begin{aligned} -\triangle u &= 0 && \text{where } u > \varphi \text{ (at the points of no stop)} \\ -\triangle u &\geq 0 && \text{everywhere in } \mathbb{R}^n \\ u &\geq \varphi \end{aligned}$$

A model like this is used in financial mathematics for pricing American options

Jump processes

We now consider

$$u(x) = \sup_{\tau} E(\varphi(X_\tau^x))$$

where X_t^x is a purely jump process starting at x and τ is any stopping time.

Then

$$\begin{aligned} Lu &= 0 && \text{where } u > \varphi \text{ (at the points of no stop)} \\ Lu &\geq 0 && \text{everywhere in } \mathbb{R}^n \\ u &\geq \varphi \end{aligned}$$

where the operator L has the integro-differential form

$$Lu(x) = \mathrm{PV} \int_{\mathbb{R}^n} (u(x) - u(x+y))K(y)\,\mathrm{d}y$$

for some positive kernel K.

Fractional laplacian

Natural example:

$$Lu(x) = \mathrm{PV}\int_{\mathbb{R}^n} \frac{u(x) - u(x+y)}{|y|^{n+2s}}\,\mathrm{d}y = C(-\triangle)^s u(x)$$

Corresponds to α-stable stochastic processes.

$$\begin{aligned} (-\triangle)^s u &= 0 && \text{where } u > \varphi \text{ (at the points of no stop)} \\ (-\triangle)^s u &\geq 0 && \text{everywhere in } \mathbb{R}^n \\ u &\geq \varphi && \end{aligned}$$

Properties of the fractional laplacian

The fractional Laplacian $(-\triangle)^s$ is a nonlocal operator that has the following simple form

$$\widehat{(-\triangle)^s u}(\xi) = |\xi|^{2s}\, \widehat{u}(\xi)$$

It also has the following properties

- Commutes with rigid motions: $(-\triangle)^s(u \circ M) = ((-\triangle)^s u) \circ M$ for any rigid motion M.
- Scales with order $2s$ in the following sense: $(-\triangle)^s u_\lambda(x) = \lambda^{2s}\,((-\triangle)^s u)\,(\lambda x)$, where $u_\lambda(x) = u(\lambda x)$.

Regularity results

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- Quasi optimal result $u \in C^{1,\alpha}$ for every $\alpha < s$ (CPAM 2005)
- Optimal regularity $u \in C^{1,s}$ (Caffarelli, Salsa, S. 2007)
- The free boundary is $C^{1,\alpha}$ except on some singular set.

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Thin Obstacle Problem - Case $s = 1/2$

$$\triangle u = 0$$

$$u(x,0) \geq \varphi(x)$$
$$\partial_{n+1} u \leq 0$$
$$\partial_{n+1} u = 0 \text{ where } u > \varphi$$

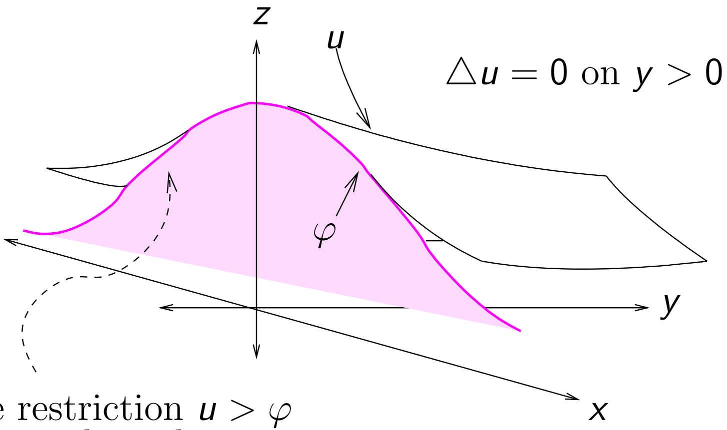
We extend the function u harmonically in the upper half space

$$\triangle u = 0 \text{ in } \{x_{n+1} > 0\},$$

then we have the relation

$$-\partial_n u(x,0) = (-\triangle)^{1/2} u(x,0)$$

A drawing of the thin obstacle problem



The restriction $u > \varphi$
only applies where $y = 0$

Obstacle problem for $(-\triangle)^{1/2}$

The thin obstacle problem in the upper half space is equivalent to the obstacle problem on the boundary with $L = (-\triangle)^{1/2}$.

$$
\begin{aligned}
u &\geq \varphi \\
(-\triangle)^{1/2} u &\geq 0 \\
(-\triangle)^{1/2} u &= 0 \qquad \text{where } u > \varphi
\end{aligned}
$$

Regularity of the thin obstacle problem

The regularity results for the thin obstacle problem are

- $u \in C^{1,\alpha}$ for a small $\alpha > 0$ (Caffarelli 1979)
- $u \in C^{1,1/2}$ (Athanasopoulos and Caffarelli, 2004)
- The free boundary is smooth under nondeg. assumptions (Athanasopoulos, Caffarelli and Salsa, 2006)

All these results would hold for the obstacle problem for $(-\triangle)^{1/2}$.

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The following properties of the Dirichlet to Neumann map can be checked directly.
Let $u(x,y)$ be the harmonic extension to the upper half plane and let

$$Lu(x,0) \mapsto -\partial_y u(x,0)$$

- L commutes with rigid motions: $L(u \circ M) = (Lu) \circ M$ for any rigid motion M.
- L scales with order 1 in the following sense: $Lu_\lambda(x) = \lambda\,(Lu)\,(\lambda x)$, where $u_\lambda(x) = u(\lambda x)$.

If we want to construct a similar extension for arbitrary fractional laplacians $(-\triangle)^s$ we have to make it scale in a different way.

Extension with a weight.

$$\operatorname{div}(y^a \nabla u) = 0$$

$$u(x,0) \geq \varphi(x)$$
$$\lim_{y \to 0} y^a \partial_{n+1} u \leq 0$$
$$\lim_{y \to 0} y^a \partial_{n+1} u = 0 \text{ where } u > \varphi$$

We extend the function u in the upper half space to satisfy the equation:

$$\operatorname{div}(y^a u) = 0 \text{ in } \{x_n > 0\},$$

then we have the relation

$$-\lim_{y \to 0} y^a \partial_n u(x,0) = c(-\triangle)^{\frac{1-a}{2}} u(x,0)$$

(Caffarelli, S., CPDE 2007)

Equivalent local problem.

An equivalent problem to the obstacle problem for $(-\triangle)^s$ is given by

$$\begin{aligned} \operatorname{div}(y^a \nabla u) &= 0 \qquad \text{in } \{y > 0\}, \\ u(x,0) &\geq \varphi(x,0), \\ \lim_{y \to 0^+} y^a u_y(x,y) &= 0 \qquad \text{in } \{u(x,0) > \varphi(x,0)\}, \\ \lim_{y \to 0^+} y^a u_y(x,y) &\leq 0 \end{aligned}$$

with $s = (1-a)/2$.

A_2 weights.

$|y|^a$ is degenerate near $y = 0$ however it is in the class of A_2 weights.

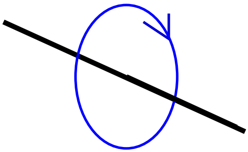
The equation $\operatorname{div} |y|^a \nabla u = 0$ satisfies:

- Harnack inequality.
- Boundary Harnack

Fabes, Kenig, Jerison, Serapioni.

Cylindrically symmetric harmonic functions

$$u(x, y) = \tilde{u}(x, |y|)$$



Let $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^{1+a}$ for some number a.

Assume u is radially symmetric in y.

Then we can express its Laplacian in cylindrical coordinates

$$\triangle u = \triangle_x u + \partial_{rr} u + \frac{a}{r} \partial_r u = r^{-a} \operatorname{div}(r^a \nabla u)$$

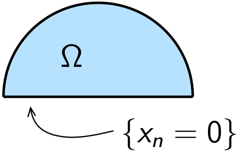
Thus the equation $\operatorname{div}(r^a \nabla u) = 0$ basicall means that u is a harmonic cylindrically symmetric function.

Fractional dimension.

Our equivalent problem can be thought as a thin obstacle problem with fractional co-dimension $1+a$.

The problem can be localized.

$$\begin{aligned} \operatorname{div}(y^a \nabla u) &= 0 && \text{in } B_1^+, \\ u(x,0) &\geq \varphi(x,0) && \text{on } B_1 \cap \{y=0\}, \\ \lim_{y\to 0^+} y^a u_y(x,y) &= 0 && \text{on } \{u(x,0) > \varphi(x,0)\} \\ \lim_{y\to 0^+} y^a u_y(x,y) &\leq 0 \end{aligned}$$



with $s = (1-a)/2$.

Similar methods to the ones employed to the classical thin obstacle problem will work.

Some tools used

- Analysis of blowup sequences.
- Almgren's frequency formula

$$\Phi(r) = r\frac{\int_{B_r^+} y^a |\nabla u|^2 \, \mathrm{d}X}{\int_{\partial B_r \cap \{y>0\}} y^a |u|^2 \, \mathrm{d}\sigma(X)} \nearrow$$

- A Liouville theorem for blowup profiles.
- Blowup profiles at nondegenerate points have a flat free boundary $\longrightarrow$ free boundary regularity.

Almgren monotonicity formula

If $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ satisfies

$$\triangle u = 0 \text{ in } \{y > 0\}$$
$$u \cdot u_y = 0 \text{ on } \{y = 0\}$$

Then the following function is monotone increasing

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Blowup sequence

Assume 0 is in the free boundary. Let

$$u_r(x) = \left(r^{-n-a} \int_{\partial B_r \cap \{y>0\}} y^a |u|^2 \, \mathrm{d}X \right)^{-1/2} u(rX)$$

There is a subsequence u_{r_j} converging to a global solution to the problem u_0.

A Liouville type theorem for blowup profiles

If $u_0 : \mathbb{R}^n \times [0, +\infty) \to \mathbb{R}$ is a homogeneous global solution to the problem

$$\begin{aligned} \operatorname{div}(y^a \nabla u) &= 0 && \text{in } \{y > 0\}, \\ u(x,0) &\geq 0, && \\ \lim_{y \to 0^+} y^a u_y(x,y) &= 0 && \text{in } \{u(x,0) > \varphi(x,0)\}, \\ \lim_{y \to 0^+} y^a u_y(x,y) &\leq 0 && \end{aligned}$$

with 0 in the free boundary, then either

- u_0 has degree $1 + s$ and the free boundary is a plane.
- u_0 has degree ≥ 2.

Optimal regularity

The degree of homogeneity of u_0 is the value $\Phi(0)$ for the Almgren frequency formula.
The fact that $\Phi(0) \geq 1 + s$ implies that $u \in C^{1,s}$.

Free boundary regularity

If $\Phi(0) = 1 + s$ then the free boundary is $C^{1,\alpha}$ in a neighborhood of the origin.

Idea:
The blowup limit u_0 has a flat free boundary. For small r, the free boundary of u_r must be arbitrarily close to flat. Once the free boundary of u_r is almost flat, a similar proof to the one of the classical obstacle problem follows using the Boundary Harnack principle.