On the regularity of a singular variational problem

Erik Lindgren
Luis Silvestre

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Abstract

We study the optimal regularity for a minimizer of a functional of the form $J(u) = \int_D \frac{|\nabla u|^2}{2} + F(u)\,\mathrm{d}x$, where F is merely Hölder continuous. Similar functionals have been studied earlier under a sign condition. Using iterative and blow-up arguments we obtain the same optimal $C^{1,\alpha}$-regularity as the known result in the case of non-negativity.

1 Introduction

Let $p \in (0,1)$, $F : \mathbb{R} \to \mathbb{R}$ be a continuous function such that it is differentiable in $\mathbb{R} \setminus \{0\}$, $F'(t) = f(t)$ and $|f(t)| \le p\Lambda |t|^{p-1}$. The function F is then only Hölder at 0. We study local minimizers of the functional

$$J(u) = \int_D \frac{|\nabla u|^2}{2} + F(u)\,\mathrm{d}x \tag{1.1}$$

This is a bounded functional in $H^1(D)$. It is easy to show that for any $g \in H^1(D)$, if we restrict J to the set $\mathcal{A} = \{v \in H^1(D) : v - g \in H^1_0(D)\}$, then J achieves a minimum in $\mathcal{A}$.

For any regularity purpose, we can assume $D = B_1$, and we will do it from now on. The corresponding Euler-Lagrange equation is

$$\triangle u = f(u) \quad \text{at least where } u \text{ is away from zero} \tag{1.2}$$

The equation can have a singularity at 0 since f can become unbounded in the origin. In this case J can never be convex. Since we lack convexity, local minimizers of (1.1) solve the equation (1.2) but the implication in the other direction does not necessarily hold. If f was assumed to be a C^α function, then it would be possible to apply standard technics to obtain that the solution u is a $C^{2,\alpha}$ function, which is optimal.

We do not assume any sign condition for u. An important special case is $F(t) = (t^+)^p$. When $p = 0$ it is the same as the two phase problem studied in [1]. The optimal regularity for the function u in that case is $C^{0,1}$. When $p = 1$, it is the same as the *two phase obstacle problem*. In that case the optimal regularity for the function u is $C^{1,1}$ as it was shown in [6] or [7] (although in the second one the hypothesis are slightly different). For both $p = 0$ and $p = 1$, the optimal regularity was achieved using the monotonicity formula developed in [1]. For the other values of p, the nonnegative case was studied in [5] and [4], and the optimal regularity was proven to coincide with the scaling of the equation, $C^{1,\beta-1}$ for $\beta = \frac{2}{2-p}$. Although this same optimal regularity would not hold for the unsigned case when $p \in (1,2)$, we will show it does when $p \in (0,1)$.

We can assume that $F(0) = 0$, since we can add constants to the functional (1.1) without altering the minimizer u. Since $|f(t)| \le p\Lambda |t|^{p-1}$, then $|F(t)| \le \Lambda |t|^p$. The main theorem of the paper is

Theorem 1.1. *A minimizer u of* (1.1) *(with $0 < p < 1$) is in $C^{1,\beta-1}(B_{1/2})$ for $\beta = \frac{2}{2-p}$ (which is the scaling of the equation and the same regularity as in the one phase case).*

Remark 1.2. For $p < 0$ this problem changes a bit, since in this case we would expect u to be merely C^α for $\alpha = \frac{2}{2-p}$. We also remark that in this case we do not only have a singularity in the pde but also in the functional. We hope to be able to treat these problems in future papers.

The equation 1.2 is a reaction diffusion equation with a singularity at zero. Since we do not assume any sign condition for u, the same theory applies for isolated singularities of f at any point. Reaction diffusion equations appear in a variety of applications including distribution of temperature in a reacting mixture, or population density in migrations models, to name a couple. The result of this paper would apply to the cases in which, for whatever reason, the function f in the equation has an isolated singularity.

2 Estimates in L^∞

Proposition 2.1. *Let u be a function in $H^1(B_1)$ solving the equation* (1.2) *in the unit ball B_1 such that $u = g$ on ∂B_1 for a continuous function g. Then $u \in L^\infty(B_1)$.*

Proof. Let $\tilde{u}(x) = \max(u(x), 1)$. Then $\triangle\tilde{u}(x) = f(u(x)) \geq -p\Lambda$ when $u(x) > 1$. Thus $\max\tilde{u} \leq \max(1, \max g) + C$ and so u is bounded above.

We can argue the same way for $\tilde{u}(x) = \min(u(x), -1)$ to obtain a bound from below for u. Thus we will have the estimate

$$||u||_{L^\infty} \leq C(n)(p\Lambda + ||g||_{L^\infty})\,. \tag{2.1}$$

□

3 Hölder regularity of the function.

We will achieve a modulus of continuity for the function u by comparing it to its harmonic replacement in a ball inside the domain B_1.

Given a ball $B \subset B_1$, we consider the function $v \in H^1(B)$ solving the following equation

$$u - v \in H_0^1(B) \tag{3.1}$$

$$\triangle v = 0. \tag{3.2}$$

We call this function v, *the harmonic replacement of u in B*.

Lemma 3.1. *Let u be a minimizer of* (1.1) *for a bounded boundary value g, then for any ball $B \subset B_1$,*

$$\int_B |\nabla(u - v)|^2 \,\mathrm{d}x \leq 4\Lambda \sup_B |u|^p\,|B| \tag{3.3}$$

Remark 3.2. By Proposition 2.1, we already know that u is bounded in B_1.

Proof. Since $u - v \in H^1(B)$ and $\triangle v = 0$ in B, then

$$\int_B |\nabla(u - v)|^2 \,\mathrm{d}x = \int_B |\nabla u|^2 - |\nabla v|^2 \,\mathrm{d}x$$

Since u is a local minimizer of J,

$$\int_B \frac{|\nabla u|^2}{2} + F(u)\,\mathrm{d}x \le \int_B \frac{|\nabla v|^2}{2} + F(v)\,\mathrm{d}x$$

By maximum principle, $\sup_B v \le \sup_B u$. Replacing in the above relations and recalling $|F(u)| \le C\,|u|^p$:

$$\int_B |\nabla(u-v)|^2\,\mathrm{d}x = \int_B |\nabla u|^2 - |\nabla v|^2 \le 4\Lambda \int_B \sup_B |u|^p\,\mathrm{d}x \le 4\Lambda \sup_B |u|^p\,|B|$$

□

Lemma 3.3. *If a bounded function $u \in H^1(B_1)$ satisfies (3.3) for any harmonic replacement v in a ball $B \subset B_1$, then u is $C^\alpha\left(B_{1/2}\right)$ for any $\alpha < 1$.*

Proof. The idea is to show an appropriate decay for the averages of $|\nabla u|^2$ of the form

$$\int_{B_{2^{-k}}(x_0)} |\nabla u|^2\,\mathrm{d}x \le C_1 2^{-k(n-\eta)} \tag{3.4}$$

for an arbitrarily small η and $x_0 \in B_{1/2}$, and then apply standard Morrey's embedding theorem.

We will show it by induction. Suppose it is true up to some value of k. Consider the harmonic replacement v in $B = B_{2^{-k}}(x_0)$. Since v is harmonic,

$$\int_B |\nabla v|^2\,\mathrm{d}x \le \int_B |\nabla u|^2\,\mathrm{d}x \le C2^{-k(n-\eta)}$$

Moreover, since v is harmonic, $|\nabla v|^2$ is subharmonic, thus

$$\int_{B_{2^{-k-1}}(x_0)} |\nabla v|^2\,\mathrm{d}x \le \frac{1}{2^n} \int_{B_{2^{-k}}(x_0)} |\nabla v|^2\,\mathrm{d}x$$

Combining the above two

$$\int_{B_{2^{-k-1}}(x_0)} |\nabla v|^2 \le \frac{1}{2^n} C_1 2^{-k(n-\eta)}$$

By hypothesis, u and v satisfy (3.3)

$$\int_{B_{2^{-k}}(x_0)} |\nabla(u-v)|^2\,\mathrm{d}x \le C2^{-kn}$$

where $C = 4\Lambda \sup |u|^p \operatorname{vol}(B_1)$.

Putting it all together we get

$$\begin{aligned}\int_{B_{2^{-k-1}}(x_0)} |\nabla u|^2\,\mathrm{d}x &\le \int_{B_{2^{-k-1}}(x_0)} |\nabla v|^2 + |\nabla(u-v)|^2 + 2\,|\nabla u|\,|\nabla v|\,\mathrm{d}x \\ &\le I_1 + I_2 + \sqrt{I_1 I_2}\end{aligned}$$

Where

$$I_1 = \int_{B_{2^{-k-1}}(x_0)} |\nabla v|^2\,\mathrm{d}x \le \frac{1}{2^n} C_1 2^{-k(n-\eta)} = 2^{-\eta} C_1 2^{-(k+1)(n-\eta)}$$

$$I_2 = \int_{B_{2^{-k-1}}(x_0)} |\nabla(u-v)|^2\,\mathrm{d}x \le C2^{-kn} = 2^{-k\eta+n-\eta} C2^{-(k+1)(n-\eta)}$$

So,

$$\int_{B_{2^{-k-1}}(x_0)} |\nabla u|^2 \, \mathrm{d}x \le \left(2^{-\eta} + \frac{C}{C_1} 2^{-k\eta+n-\eta} + \sqrt{\frac{C}{C_1}} 2^{(-k\eta+n-\eta)/2}\right) C_1 2^{-(k+1)(n-\eta)}$$
$$\le C_1 2^{-(k+1)(n-\eta)}$$

as long as $\frac{C}{C_1}$ is small enough. Notice that the value of C_1 for which this happens depends only on Λ, η, n and $\|u\|_{L^\infty}$.

This finishes the proof of (3.4). Now this implies that $u \in C^\alpha$ for any $\alpha < 1$ by the classical Morrey's embedding (which can be found for example in [3], Theorem 7.19). □

Corollary 3.4. *The minimizer u of* (1.1) *is in the class $C^\alpha(B_{1/2})$ for any $\alpha < 1$. Moreover*

$$[u]_{C^\alpha(B_{1/2})} \le C(\eta, n) p\Lambda (p\Lambda + ||g||_{L^\infty})^p \,. \tag{3.5}$$

Proof. We can take $C_1 \le C(\eta, n) p\Lambda \sup |u|^p$. This together with (2.1) yields (3.5). □

4 Hölder regularity of the derivatives.

To prove a $C^{1,\alpha}$ estimate we will proceed in a similar fashion as in section 3. But our iteration has to be more careful and it is only going to work for small values of α. We will also use this as a way to show Lipschitz continuity. We could also achieve a uniform Lipschitz bound using Alt-Caffarelli-Friedman monotonicity formula. We will not need to do this because we are assuming $p > 0$ (although the estimate blows up as $p \to 0^+$).

Lemma 4.1. *If v is a harmonic function in a ball $B_r(x_0)$, then for a small enough $\sigma > 0$.*

$$\int_{B_{\sigma r}(x_0)} |\nabla v - \nabla v(x_0)|^2 \, \mathrm{d}x \le (1-\theta)\sigma^n \int_{B_r(x_0)} |\nabla v|^2 \, \mathrm{d}x$$

where $\theta \in (0,1)$.

Proof. This just follows from the fact that since v is harmonic, then it has all kinds of estimates. In particular we can estimate its $C^{1,1}$ norm in $B_{r/2}$ from $\int_{B_r(x_0)} |\nabla v|^2 \, \mathrm{d}x$. Namely

$$\left|D^2 v(x)\right| \le C r^{-n/2-1} \left(\int_{B_r(x_0)} |\nabla v|^2 \, \mathrm{d}x\right)^{1/2}$$

then for any $x \in B_{\sigma r}(x_0)$,

$$|\nabla v(x) - \nabla v(0)|^2 \le C r^{-n-2} \left(\int_{B_r(x)} |\nabla v|^2 \, \mathrm{d}x\right) (\sigma r)^2$$

Integrating we obtain

$$\int_{B_{\sigma r}(x_0)} |\nabla v - \nabla v(x_0)|^2 \, \mathrm{d}x \le C r^{-n-2} \left(\int_{B_r(x)} |\nabla v|^2 \, \mathrm{d}x\right) (\sigma r)^2 (\sigma r)^n$$
$$\le C\sigma^2 \sigma^n \left(\int_{B_r(x)} |\nabla v|^2 \, \mathrm{d}x\right)$$

Now, for any $\theta \in (0,1)$, we can make $C\sigma^2 < 1-\theta$ if we choose σ small enough. □

Theorem 4.2. *A minimizer u of* (1.1) *is $C^{1,\alpha}(B_{1/2})$ for a small $\alpha > 0$. There is an upper bound for $\|u\|_{C^{1,\alpha}(B_{1/2})}$ that depends on Λ, p, $\|u\|_{L^\infty}$, α and the dimension n.*

Proof. The idea is like in the proof of Lemma 3.3, but this time we want to show that for each $x_0 \in B_{1/2}$, there is a vector $A(x_0)$ (which will turn out to be $\nabla u(x_0)$) such that we have the following

$$\int_{B_r(x_0)} |\nabla u - A(x_0)|^2 \, \mathrm{d}x \leq C_0 r^{n+\eta} \tag{4.1}$$

for some small value $\eta > 0$ and any $r < 1/2$. Then $C^{1,\alpha}$ regularity follows from a result of Campanato [2] with $\alpha = \eta/2$.

We will also do it iteratively, but instead of using balls of radius $(1/2)^j$, we will use σ^j as the radius, for the σ of Lemma 4.1. Our choice of C_0 will depend only on Λ, p, $\|u\|_{L^\infty}$, α and the dimension n.

For each $x_0 \in B_{1/2}$, we will iteratively construct a sequence A_j such that

$$\int_{B_{\sigma^j}(x_0)} |\nabla u - A_j|^2 \, \mathrm{d}x \leq C_1 \sigma^{j(n+\eta)} \tag{4.2}$$

$$|A_j - A_{j+1}| \leq C_2 \sigma^{j\eta/2} \tag{4.3}$$

But this iteration will continue only as long as $\inf_{B_{\sigma^j}(x_0)} |u| \leq \sigma^j$. In the the other case, equation (1.2) would be nondegenerate in $B_{\sigma^j}(x_0)$, and we would be able to apply $C^{1,2\eta}$ estimates to obtain (4.1) for $r \leq \sigma^{j+1}$ and $A(x_0) = \nabla u(x_0)$, and there would be no need to continue the iteration. In case the iteration continues forever, we would define $A(x_0) = \lim A_j$ and we will obtain (4.1) from (4.2). In any case it will hold $A_j - A(x_0) \leq C\sigma^{j\eta/2}$.

Let us first show that (4.2) and (4.3) hold as long as we have $\inf_{B_{\sigma^j}(x_0)} |u| \leq \sigma^j$ for every $j \leq k$. The proof is by induction. We can choose C_1 and C_2 large enough so that the statement is true for $j = 1$, we want to check that the inductive iteration holds. We assume (4.2) and (4.3) hold for $j = k$ and also $\inf_{B_{\sigma^k}(x_0)} |u| \leq \sigma^k$. We will show that then (4.2) and (4.3) hold for $j = k + 1$.

Consider the harmonic replacement v of u in $B = B_{\sigma^k}(x_0)$. Actually, we see that $v - A_k \cdot x$ is the harmonic replacement of $u - A_k \cdot x$ in B. Therefore

$$\int_{B_{\sigma^k}(x_0)} |\nabla v - A_k|^2 \, \mathrm{d}x \leq \int_{B_{\sigma^k}(x_0)} |\nabla u - A_k|^2 \, \mathrm{d}x =: I_1 \tag{4.4}$$

We set $A_{k+1} = \nabla v(x_0)$. By Lemma 4.1 applied to $v - A_k \cdot x$, we have

$$\int_{B_{\sigma^{k+1}}(x_0)} |\nabla v - A_{k+1}|^2 \, \mathrm{d}x \leq (1-\theta)\sigma^n \int_{B_{\sigma^k}(x_0)} |\nabla v - A_k|^2 \, \mathrm{d}x \leq \theta \sigma^n I_1$$

Since we are assuming $\inf_{B_{\sigma^k}(x_0)} |u| \leq \sigma^k$, we can choose any $\beta \in (0,1)$ and $u \in C^\beta$, then $\sup_{B_{\sigma^k}(x_0)} |u| \leq C\sigma^{\beta k}$. By Lemma 3.1,

$$I_2 := \int_{B_{\sigma^k}(x_0)} |\nabla u - \nabla v|^2 \, \mathrm{d}x \leq 4\Lambda \sup_{B_{\sigma^k}} |u|^p \, |B_{\sigma^k}| \leq C\sigma^{k(\beta p + n)}$$

We choose η small enough such that $\sigma^\eta > 1 - \frac{\theta}{2}$ and $\eta < \beta p$ (recall that β was actually chosen arbitrarily and it is any number less than one). As in the proof of Proposition 3.3, we have

$$\begin{aligned}
\int_{B_{\sigma^{k+1}}(x_0)} |\nabla u - A_{k+1}|^2 \, \mathrm{d}x &\le (1-\theta)\sigma^n I_1 + I_2 + \sqrt{\theta \sigma^n I_1 I_2} \\
&\le (1-\theta)\sigma^n C_1 \sigma^{k(n+\eta)} + C\sigma^{k(\beta p+n)} + \sqrt{\theta C_1 C}\sigma^{\frac{n+kn+k\eta+k\beta p+kn}{2}} \\
&\le C_1 \sigma^{(k+1)(n+\eta)} \left(1 - \frac{\theta}{2-\theta} + \frac{C}{C_1}\sigma^{k(\beta p-\eta)-n-\eta} + \sqrt{\frac{C}{C_1}}\sigma^{(k(\beta p-\eta)-n-\eta)/2}\right) \\
&\le C_1 \sigma^{(k+1)(n+\eta)}
\end{aligned}$$

as long as $\frac{C}{C_1}$ is small enough.

This shows (4.2) for $j = k+1$. Note that we did not use (4.3) in the iteration for (4.2). Now we can obtain (4.3) for $j = k+1$ using (4.2) and C^1 estimates for the harmonic function v. Since $A_{k+1} - A_k$ is the gradient of $v - A_k \cdot x$ at zero, then

$$\begin{aligned}
|A_{k+1} - A_k| &\le \frac{C}{\sigma^{kn/2}} \left(\int_{B_{\sigma^k}(x_0)} |\nabla v - A_k|^2 \, \mathrm{d}x \right)^{1/2} \\
&\le C C_1^{1/2} \sigma^{k\eta/2} = C_2 \sigma^{k\eta/2}
\end{aligned}$$

Notice that (4.3) implies that $|A_k - A_j| \le C\sigma^{k\eta/2}$ for any $j > k$. If the iteration goes on forever, then A_k converges, and we immediately have (4.1) for $A(x_0) = \lim A_k$ if C_0 is large enough.

If the iteration stops at one step k, that means that $\inf_{B_{\sigma^k}(x_0)} |u| > \sigma^k$, then from (1.2), $\triangle u$ is bounded (recall $0 < p \le 1$)

$$0 \le \triangle u \le p\sigma^{k(p-1)}$$

Therefore, we can apply $C^{1,\alpha}$ estimates for $u - A_k \cdot x$ (notice $\triangle(u - A_k \cdot x) = \triangle u$), for $r = \sigma^k$ and $\alpha = \eta/2$ we have

$$\begin{aligned}
|\nabla u(x_0) - A_k| &\le Cr \|\triangle u\|_{L^\infty(B_r(x_0))} + r^{-n/2} \left(\int_{B_r(x_0)} |\nabla u - A_k|^2 \, \mathrm{d}x \right)^{1/2} \\
&\le Cp\sigma^{kp} + \sqrt{C_1}\sigma^{k\eta/2} \\
&\le C\sigma^{k\eta/2} \quad \text{as long as } p > \eta/2 \\
[\nabla u - A_k]_{C^\alpha(B_{r/2}(x_0))} &\le Cr^{1-\alpha} \|\triangle u\|_{L^\infty(B_r(x_0))} + r^{-n/2-\alpha} \left(\int_{B_r(x_0)} |\nabla u - A_k|^2 \, \mathrm{d}x \right)^{1/2} \\
&\le Cp\sigma^{k(p-\alpha)} + \sqrt{C_1}\sigma^{k(\eta/2-\alpha)} \\
&\le C_3 \quad \text{as long as } p > \eta/2
\end{aligned}$$

Now we set $A(x_0) = \nabla u(x_0)$, for any $r \le \sigma^{k+1}$, we integrate the above estimate to obtain

$$\begin{aligned}
\int_{B_r(x_0)} |\nabla u - A(x_0)|^2 \, \mathrm{d}x &\le \int_{B_r(x_0)} C_3 |x - x_0|^{2\alpha} \, \mathrm{d}x \\
&\le C_3 |B_1| r^{n+\eta}
\end{aligned}$$

So, setting $C_0 \ge C_3 |B_1|$, we obtain (4.1) for all $r \le \sigma^{k+1}$.

The fact that $|A_j - A(x_0)| \leq C\sigma^{k\eta/2}$ follows from $|A_k - A(x_0)| \leq C\sigma^{k\eta/2}$ and (4.3). This, together with (4.2) imply (4.1) for $r \geq \sigma^{k+1}$ by choosing C_0 large.

Finally, using Campanato's result [2], we obtain $\nabla u \in C^{\eta/2}$. Since C_1 is to be chosen such that C/C_1 is small where C is the constant from Lemma 3.1 we remark that we have the following estimate

$$[u]_{C^{1,\alpha}}(B_{1/2}) \leq C(p,n,\Lambda)(p\Lambda + ||g||_{L^\infty})^p\,. \tag{4.5}$$

□

We can also scale the above theorem to obtain a version in B_r.

Corollary 4.3. *A minimizer u of (1.1) in B_r such that $\|u\|_{L^\infty} \leq M$ is $C^{1,\alpha}(B_{r/2})$ for a small $\alpha > 0$. There is an upper bound for $\|u\|_{C^{1,\alpha}(B_{r/2})}$ of the form*

$$[u]_{C^{1,\alpha}(B_{r/2})} \leq r^{\beta-1-\alpha} C(r^{-\beta}M)$$

Which also implies the estimate for the Lipschitz norm

$$[u]_{C^{0,1}(B_{r/2})} \leq r^{\beta-1} C(r^{-\beta}M)$$

Where $\beta = \frac{2}{2-p}$ and C is an increasing function depending on Λ, n, p and α.

Proof. We see that $u_r(x) = r^{-\beta}u(rx)$ is a minimizer of (1.1) in B_1, so we can apply Theorem 4.2 to u_r to get the result. □

5 When the derivatives are bounded below.

In this section we will show that if $|\nabla u|$ is bounded below in B_1, then $u \in C^{1,p}(B_{1/2})$, which is better than optimal. The norm will naturally depend on the lower bound on $|\nabla u|$.

Lemma 5.1. *Let u be a C^1 function in $\overline{B_1}$ such that $a \leq |\nabla u| \leq A$. Then for any ball $B_r(x_0)$ included in B_1, the following estimate holds*

$$|\{-\lambda < u < \lambda\} \cap B_r(x_0)| \leq Cr^{n-1}\lambda \tag{5.1}$$

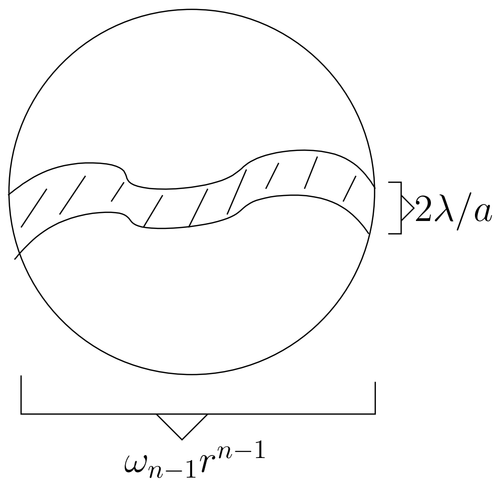
for a constant C that depends only on dimension, a, A, and the modulus of continuity of ∇u.

Proof. Since $u \in C^1(\overline{B_1})$, for a small enough $r_0 > 0$ (depending only on a and the modulus of continuity for ∇u),

$$\operatorname*{osc}_{B_{r_0}(x)\cap B_1} \langle \nabla u, e\rangle \leq \frac{a}{2}$$

for any $x \in B_1$ and unit vector e.

Let $e = \frac{\nabla u(x_0)}{|\nabla u(x_0)|}$. The above relation implies that $u_e \geq \frac{a}{2}$ in the ball $B_{r_0}(x_0)$. This means that u is strictly increasing in e, therefore if we look at e as the direction that points up, all the level sets of u will be the graph of some function. Moreover, since $u_e \geq \frac{a}{2}$, then $\{u = \lambda\}$ and $\{u = -\lambda\}$ will be at distance at most $\frac{4\lambda}{a}$ in the direction of e. Thinking of both level sets as graphs of functions that means that the corresponding functions differ by at most $\frac{4\lambda}{a}$. If r is any radius less than r_0, then the same thing applies and the measure of the set $|\{-\lambda < u < \lambda\} \cap B_r(x_0)|$ has to be less or equal than $\omega_{n-1}\frac{4\lambda}{a}r^{n-1}$, where ω_{n-1} is the volume of the $n-1$-dimensional sphere.



If on the other hand $r > r_0$, then we cover $B_r(x_0)$ with balls of radius r_0 and in each piece we apply the above reasoning. We obtain

$$\begin{aligned}|\{-\lambda < u < \lambda\} \cap B_r(x_0)| &\leq N\omega_{n-1}\frac{4\lambda}{a}r_0^{n-1}\lambda \\ &\leq N\omega_{n-1}\frac{4\lambda}{a}r^{n-1}\lambda\end{aligned}$$

where N is the number of balls of radius r_0 that we need to cover B_r. But N is bounded by the number of balls of radius r_0 that we would need to cover the whole B_1, that is a fixed number depending only on dimension and r_0. □

Remark 5.2. Looking at the proof of Lemma 5.1, it may seem that the constant C does not depend on A. That is somewhat misleading because A is implicit in the modulus of continuity for ∇u if $|\nabla u| = a$ was actually achieved.

Remark 5.3. It is not clear whether the constant of Lemma 5.1 should or should not depend on the modulus of continuity but only on a and A. We leave it as an interestin question. Of course, for our proof to work the constant must depend on the modulus of continuity.

Proposition 5.4. *Let u be a minimizer of* (1.1) *(with $0 < p < 1$) in B_1 such that $\|u\|_{L^\infty(B_1)} \leq M$ and $|\nabla u| \geq a$, then $u \in C^{1,p}(B_{1/2})$. Moreover, there is an estimate of the form*

$$[u]_{C^{1,p}(B_{1/4})} \leq C(M, a) \tag{5.2}$$

where $C(M, a)$ is some function of M and a that depends also on dimension.

Proof. We apply Theorem 4.2 to obtain that $u \in C^{1,\alpha}(B_{1/2})$. In particular $u \in C^1(B_{1/2})$ with a C^α modulus of continuity for ∇u (depending on M) and $A := \sup|\nabla u| \leq C(M)$. Then we can apply Lemma 5.1 to a rescaling of u to obtain

$$|\{-\lambda < u < \lambda\} \cap B_r(x_0)| \leq Cr^{n-1}\lambda$$

We want to use (1.2) to control the behavior of $\triangle u$. First of all we must notice that since $u \in C^{1,\alpha}$ and $|\nabla u| > a$, by the implicit function theorem $\{u = 0\}$ is a C^1 surface. Moreover, since $u \in C^{1,\alpha}$, there is no jump of the derivative across this surface, and therefore $\triangle u$ has no singular part on $\{u = 0\}$.

Recalling that u solves (1.2), we obtain

$$|\{|\triangle u| > \lambda\} \cap B_r(x_0)| \leq C r^{n-1} \lambda^{\frac{1}{1-p}}$$

for any ball $B_r \subset B_{1/2}$. Then

$$\int_{B_r} |\triangle u| \, dx \leq \int_0^\infty |\{|\triangle u| > \lambda\} \cap B_r(x_0)| \, d\lambda \leq C r^{n-1+p} \tag{5.3}$$

which implies (5.2) as shown in the appendix. □

Corollary 5.5. *With the same hypotheses of Proposition 5.4, we have*

$$[u]_{C^{1,\beta-1}(B_{1/4})} \leq C(M, a) \tag{5.4}$$

where $\beta = \frac{2}{2-p}$.

Proof. $\beta \leq 1 + p$ □

Corollary 5.6. *Let u be a minimizer of* (1.1) *(with $0 < p < 1$) in B_1 such that $\|u\|_{L^\infty(B_r)} \leq M$ and $|\nabla u| \geq a$, then $u \in C^{1,\beta-1}(B_{1/2})$. Moreover, there is an estimate of the form*

$$[u]_{C^{1,\beta-1}(B_{r/4})} \leq C(r^{-\beta} M, \frac{a}{r^{\beta-1}}) \tag{5.5}$$

Proof. We see that $u_r(x) = r^{-\beta} u(rx)$ is a minimizer of (1.1) in B_1, so we can apply Corollary 5.5 to u_r to get (5.5). □

6 Optimal regularity for $p \in (0, 1)$

We will prove that when $p \in (0, 1)$ then the optimal regularity of the minimizers of (1.1) is $C^{1,\beta-1}$ for $\beta = \frac{2}{2-p}$, which comes from the scaling of the equation and the same as the optimal regularity for the nonnegative case when $F(u) = u^p$ (see [5]).

The following lemma exploits the scaling of the equation via a blowup argument.

Lemma 6.1. *Let u be a minimizer of* (1.1) *in B_1 such that $\|u\|_{L^\infty(B_1)} \leq M$. Then there is a constant C, depending only on p, M, and dimension, such that if $r < 1/2$ and $\beta = \frac{2}{2-p}$, one of the following happens*

1. $\inf_{B_r} u \geq r^\beta$
2. $\inf_{B_r} |\nabla u| \geq r^{\beta-1}$
3. $\sup_{B_r} |u| \leq C r^\beta$
4. $\sup_{B_r} |u| \leq 2^{-j\beta} \sup_{B_{2^j r}} |u|$ *for some $j \geq 1$ such that $2^j r \leq 1$.*

Proof. Suppose there is no such constant C. Then for every $t > 1$ we would be able to find a u_t and r_t such that $\|u_t\|_{L^\infty} \leq M$ and all of the following hold

1. $\inf_{B_{r_t}} u_t \leq r_t^\beta$

2. $\inf_{B_{r_t}} |\nabla u_t| \leq r_t^{\beta-1}$

3. $\sup_{B_{r_t}} |u_t| \geq t r_t^{\beta}$

4. $\sup_{B_{r_t}} |u_t| \geq 2^{-j\beta} \sup_{B_{2^j r_t}} |u_t|$ for every $j \geq 1$ such that $2^j r_t \leq 1$.

For (3) to hold, r_t must go to zero as $t \to \infty$ because the functions u_t are bounded uniformly. If we consider

$$\tilde{u}_t = \frac{1}{\sup_{B_{r_t}} |u_t|} u_t(r_t x)$$

then $\tilde{u}_t$ is a local minimizer of the functional

$$J_t(v) := \int |\nabla v|^2 + F_t(v) \, \mathrm{d}x$$

where $F_t(v) = \frac{r_t^2}{(\sup_{B_{r_t}} |u_t|)^2} F\left(\sup_{B_{r_t}} |u_t| \, v\right)$ satisfies $|F_t(v)| \leq \left(\frac{r_t^{\beta}}{\sup_{B_{r_t}} |u_t|^2}\right) \Lambda |v|^p$ that goes to zero as $t \to \infty$ because of (3). Moreover, for $\tilde{u}_t$ all of the following hold

1. $\inf_{B_1} \tilde{u}_t \leq t^{-2}$

2. $\inf_{B_1} |\nabla \tilde{u}_t| \leq t^{-2}$

3. $\sup_{B_1} |\tilde{u}_t| = 1$

4. $\sup_{B_{2^j}} |\tilde{u}_t| \leq 2^{j\beta}$ for every $j \geq 1$ such that $2^j \leq \frac{1}{r_t}$.

For $j < 1$ (which holds for $t > 1$), we have a uniform $C^{1,\alpha}$ estimate for $\tilde{u}_t$ for a small α. This means that we can extract a subsequence such that $\tilde{u}_t$ and $\nabla \tilde{u}_t$ converge uniformly to some function u_∞ and ∇u_∞ respectively. Then function u_∞ has to be a local minimizer of

$$J_\infty(v) := \int |\nabla v|^2 \, \mathrm{d}x$$

But this means that u_∞ is harmonic and satisfies

1. $\inf_{B_1} u_\infty \leq 0$

2. $\inf_{B_1} |\nabla u_\infty| \leq 0$

3. $\sup_{B_1} |u_\infty| = 1$

4. $\sup_{B_{2^j}} |u_\infty| \leq 2^{j\beta}$ for every $j \geq 1$ such that $2^j \leq \frac{1}{r_t}$.

From (4), u_∞ must be of the form $ax + b$ since it is a harmonic function that grows less than quadratic at infinity. From (2), $a = 0$, and then from (1), $b = 0$. But then $u_\infty \equiv 0$ which contradicts (3). □

Theorem 6.2. *A minimizer u of* (1.1) *(with $0 < p < 1$) is in $C^{1,\beta-1}(B_{1/2})$ for $\beta = \frac{2}{2-p}$ (which is the scaling of the equation and the same regularity as in the one phase case).*

Proof. The proof follows more or less a similar strategy as in Theorem 4.2. We will prove some decay by iterating Lemma 6.1 that this time will work as long as u and $|\nabla u|$ remain small. When they are too large we apply either the estimates for a function with bounded laplacian, or Proposition 5.4.

For any $x_0 \in B_{1/2}$, we apply iteratively Lemma 6.1 for ball of radius $r = 2^{-j}$ centered in x_0 for as long as we have

$$\sup_{B_{2^{-j}}} u \leq 2^{-j\beta}$$
$$\sup_{B_{2^{-j}}} |\nabla u| \leq 2^{-j(\beta-1)}$$

in case we can carry out the iteration forever, we have $\nabla u(x_0) = 0$ and

$$\sup_{B_{2^{-j}}(x_0)} |u| \leq C 2^{-j\beta} \quad \text{for any } j$$

which means that $|u(x)| \leq C|x - x_0|^\beta$ for $x \in B_{1/2}$ and u is $C^{1,\beta-1}$ at x_0.

In case the iteration stop for some step $j = k$, we would have

$$\sup_{B_{2^{-j}}(x_0)} |u| \leq C 2^{-j\beta} \quad \text{for } j \leq k$$

which means that there is a constant C for which

$$|u(x)| \leq C|x - x_0|^\beta \text{ for } x \in B_{1/2} \setminus B_{r/4}(x_0) \tag{6.1}$$

for $r = 2^{-k}$ and

$$\sup_{B_r(x_0)} |u| \leq Cr^\beta \tag{6.2}$$

If the iteration stopped at $j = k$ it is because (for $r = 2^{-k}$) either

$$\inf_{B_r(x_0)} u \geq r^\beta$$

or

$$\inf_{B_r(x_0)} |\nabla u| \geq r^{\beta-1}$$

We have to analyze both cases.

Case 1. If $\inf_{B_r} |u| \geq r^\beta$, then $\triangle u$ is bounded in $B_r(x_0)$ by

$$|\triangle u(x)| \leq pr^{(p-1)\beta}$$

Then u has $C^{1,\beta-1}$ estimates in $B_{r/2}$. Using (6.2), they give

$$\begin{aligned} |\nabla u(x_0)| &\leq \frac{C}{r} \sup_{B_r(x_0)} |u| + Cr \sup_{B_r(x_0)} |\triangle u| \\ &\leq Cr^{\beta-1} \end{aligned}$$

$$\begin{aligned} [u]_{C^{1,\beta-1}(B_{r/2})} &\leq \frac{C}{r^\beta} \sup_{B_r(x_0)} |u| + Cr^{2-\beta} \sup_{B_r(x_0)} |\triangle u| \\ &\leq C \end{aligned}$$

Therefore $|u(x) - u(x_0) - (x - x_0) \cdot \nabla u(x_0)| \leq C |x - x_0|^\beta$ for $x \in B_{r/2}(x_0)$. On the other hand for $x \in B_{1/2} \setminus B_{r/2}(x_0)$ we use (6.2) with the bound on $|\nabla u|$,

$$\begin{aligned}|u(x) - u(x_0) - (x - x_0) \cdot \nabla u(x_0)| &\leq |u(x) - u(x_0)| + |x - x_0| \, |\nabla u(x_0)| \\ &\leq C(|x - x_0|^\beta + |x - x_0| \, r^{\beta-1}) \leq C |x - x_0|^\beta\end{aligned}$$

Then u is $C^{1,\beta-1}$ at x_0.

Case 2. If $\inf_{B_r(x_0)} |\nabla u| \geq r^{\beta-1}$, then we can apply Corollary 5.6 to u in $B_r(x_0)$ to obtain the following $C^{1,\beta-1}$ estimate in $B_{r/4}(x_0)$ of the form

$$[u]_{C^{1,\beta-1}(B_{r/4})} \leq C \tag{6.3}$$

for a constant C that depends only on dimension. We can also apply Corollary 4.3 with (6.2) to obtain

$$|\nabla u(x_0)| \leq r^{\beta-1} C \tag{6.4}$$

From (6.3) we have that $|u(x) - u(x_0) - (x - x_0) \cdot \nabla u(x_0)| \leq C |x - x_0|^\beta$ for $x \in B_{r/4}(x_0)$. On the other hand when $x \in B_{1/2} \setminus B_{r/4}(x_0)$ we can do exactly as before combining (6.4) with (6.1),

$$\begin{aligned}|u(x) - u(x_0) - (x - x_0) \cdot \nabla u(x_0)| &\leq |u(x) - u(x_0)| + |x - x_0| \, |\nabla u(x_0)| \\ &\leq C(|x - x_0|^\beta + |x - x_0| \, r^{\beta-1}) \leq C |x - x_0|^\beta\end{aligned}$$

and we obtain that u is $C^{1,\beta-1}$ at x_0.

Since none of the constants C along the proof depend on x_0, we have that $u \in C^{1,\beta-1}(B_{1/2})$. □

Remark 6.3. For $p = 1$, our proof does not work for two reasons. The first one is that estimate (5.3) does not imply that $u \in C^{1,1}$. The second, and maybe most important, is that in the proof of lemma 6.1 we would have our limit function growing quadratically, and therefore it is not necessarily a plane.

The optimal regularity for $p = 1$ was proven to be $C^{1,1}$ when f' is bounded below very recently using ACF monotonicity formula in [6].

7 Appendix: Proof that 5.3 implies $C^{1,p}$

In this appendix we show the following result that we need for section 5

Theorem 7.1. *Let $u : B_1 \to \mathbb{R}$ be a bounded function such that*

$$\begin{aligned} &u \leq M && \text{in } B_1 \\ &\int_{B_r(x)} |\triangle u| \; \mathrm{d}y \leq M r^{n-1+p} && \text{for any ball } B_r(x) \subset B_1 \end{aligned}$$

then $u \in C^{1,p}(B_{1/4})$. Moreover

$$\nabla u(x_1) - \nabla u(x_2) \leq CM |x_1 - x_2|^p$$

for any $x_1, x_2 \in B_{1/4}$, where the constant C depends only on p and the dimension n.

Remark 7.2. By a standard covering argument, the theorem implies that the same result is true if we replace $B_{1/4}$ for any other set compactly contained in B_1. In that case, the constant C would depend on that set too.

Proof. We can prove it assuming that u is smooth. A density argument extends it to any bounded function satisfying the hypothesis.

Let us consider

$$\begin{aligned} u &= u_1 + u_2 \\ u_1(x) &= \int_{B_{1/3}} \frac{C_n}{|x-y|^{n-2}} \triangle u(y)\, \mathrm{d}y \\ \triangle u_2(x) &= 0 \quad \text{in } B_{1/3}(z) \end{aligned}$$

First, we estimate $|u_1(x)|$ for $x \in B_{1/3}$,

$$\begin{aligned} |u_1(x)| &= \left| \int_{B_{1/3}} \frac{C_n}{|x-y|^{n-2}} \triangle u(y)\, \mathrm{d}y \right| \\ &\leq \int_{B_{1/3}(z)} \frac{C_n}{|x-y|^{n-2}} |\triangle u(y)|\, \mathrm{d}y \leq \int_{B_{2/3}(x)} \frac{C_n}{|x-y|^{n-2}} |\triangle u(y)|\, \mathrm{d}y \\ &\leq C_n \left(2^{n-2} \int_{B_{2/3}(x)} |\triangle u(y)|\, \mathrm{d}y + \int_0^{2/3} (n-2)\rho^{-n+1} \int_{B_\rho(x)} |\triangle u(y)|\, \mathrm{d}y\, \mathrm{d}\rho \right) \\ &\leq C_{n,p} M \end{aligned}$$

Since $\triangle u_1 = 0$ outside $B_{1/3}$ and u_1 vanishes at infinity, $u_1 \leq C_{n,p} M$ everywhere.

Since $u_2 = u - u_1$, then $u_2 \leq CM$ in B_1. Moreover, since u_2 is harmonic in $B_{1/3}(z)$, then

$$|\nabla u_2(x_1) - \nabla u_2(x_2)| \leq CM\, |x_1 - x_2|^p \tag{7.1}$$

for $x_1, x_2 \in B_{1/4}$.

Second, we estimate $|\nabla u_1(x_1) - \nabla u_1(x_2)|$ for $x_1, x_2 \in B_{1/4}$. Let $z = \frac{x_1+x_2}{2}$, and $R = |x_1 - x_2|$. Recall

$$\nabla u_1(x) = \int_{B_{1/3}} C_n \frac{x-y}{|x-y|^n} \triangle u(y)\, \mathrm{d}y$$

Now we write $\nabla u_1(x_1) - \nabla u_1(x_2) = I_1 + I_2$ where

$$\begin{aligned} I_1 &= \int_{B_R(z)} C_n \left(\frac{x_1-y}{|x_1-y|^n} - \frac{x_2-y}{|x_2-y|^n} \right) \triangle u(y)\, \mathrm{d}y \\ I_2 &= \int_{B_{1/3}(z)\setminus B_R(z)} C_n \left(\frac{x_1-y}{|x_1-y|^n} - \frac{x_2-y}{|x_2-y|^n} \right) \triangle u(y)\, \mathrm{d}y \end{aligned}$$

For I_1 we do

$$\begin{aligned} |I_1| &\leq \sum_{i=1,2} \left| \int_{B_R(z)} C_n \frac{x_i-y}{|x_i-y|^n} \triangle u(y)\, \mathrm{d}y \right| \\ &\leq C_n \sum_{i=1,2} \left(\frac{1}{(2R)^{n-1}} \int_{B_{2R}(x_i)} |\triangle u(y)|\, \mathrm{d}y + \int_0^{2R} (n-1)\rho^{-n} \int_{B_\rho(x_i)} |\triangle u(y)|\, \mathrm{d}y\, \mathrm{d}\rho \right) \\ &\leq C_{n,p} M R^p \end{aligned}$$

For I_2, we must take cancellations into account, but we can extend the domain of integration to $\mathbb{R}^n \setminus B_R(z)$.

$$\begin{aligned} |I_2| &\leq \int_{\mathbb{R}^n \setminus B_R(z)} C_n \, |x_1 - x_2| \, \frac{1}{|y-z|^n} \, |\triangle u(y)| \, \mathrm{d}y \\ &\leq C_n R \int_{2R}^{\infty} n\rho^{-n-1} \int_{B_\rho(z)} |\triangle u(y)| \, \mathrm{d}y \, \mathrm{d}\rho \\ &\leq C_{n,p} M R^p \end{aligned}$$

Thus $|\nabla u_1(x_1) - \nabla u_1(x_2)| \leq |I_1| + |I_2| \leq C_{n,p} M R^p$. Combining this with (7.1), we obtain

$$|\nabla u(x_1) - \nabla u(x_2)| \leq CM \, |x_1 - x_2|^p$$

□

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Luis Silvestre
Courant Institute of Mathematical Sciences
silvestr@cims.nyu.edu

Erik Lindgren
Kungliga Tekniska högskolan
eriklin@kth.se

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