$C^{1,\alpha}$ regularity for the homogeneous parabolic p-Laplace equation

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Outline

The p-Laplace operator

Let $p \in (0, \infty)$. The p-Laplace equation arises as the Euler-Lagrange equation of the functional

$$F(u) := \int |\nabla u|^p \, \mathrm{d}x.$$

A function is p-Harmonic when

$$\triangle_p u = \mathrm{div}[|\nabla u|^{p-2} \nabla u] = 0.$$

It is a classical result that p-harmonic functions are $C^{1,\alpha}$ for some $\alpha > 0$. The optimal value of α depends on p and dimension and it is currently unknown in general.
Uraltseva [1968, $p \geq 2$], Uhlenbeck [1977 - systems - $p \geq 2$], Evans [1982, $p \geq 2$], DiBenedetto [1983], Lewis [1983], Tolksdorf [1984] and Wang [1994].

The gradient flow equation

The following parabolic p-Laplace equation is the gradient flow of the functional $\int |\nabla u|^p \, \mathrm{d}x$.

$$u_t = \mathrm{div}\left[|\nabla u|^{p-2}\nabla u\right].$$

The solutions are also known to be $C^{1,\alpha}$ in space for some $\alpha > 0$. This was proved by DiBenedetto and Friedman [1985] and Wiegner [1986] (some extra conditions are needed for $p \in (1,2)$).

Non-divergence version of the p-Laplacian

Let us expand the formula of the p-Laplacian.

$$\begin{aligned}\triangle_p u &= \operatorname{div}\left[|\nabla u|^{p-2}\nabla u\right], \\ &= |\nabla u|^{p-2}\left(\triangle u + (p-2)\frac{\partial_i u \partial_j u}{|\nabla u|^2}\partial_{ij} u\right).\end{aligned}$$

Therefore, the elliptic equation $\triangle_p u = 0$ is equivalent to

$$\triangle u + (p-2)\frac{\partial_i u \partial_j u}{|\nabla u|^2}\partial_{ij} u = 0.$$

The ∞-Laplacian

The cases $p = 1, \infty$ are special. They are best understood in non divergence form.
As $p \to \infty$, the equation converges to

$$\partial_i u \ \partial_j u \ \partial_{ij} u = 0.$$

Solutions to this are ∞-Harmonic functions. They correspond to optimal Lipschitz extensions. They are known to be $C^{1,\alpha}$ in 2D (Evans and Savin [2008]) and pointwise differentiable in arbitrary dimension (Evans and Smart [2011]). They are conjectured to be $C^{1+1/3}$.
The ∞-hamonic functions also correspond to the value function of the **stochastic "tug of war" game** (Peres, Schramm, Sheffield and Wilson [2009]). At the discrete level, this is a similar construction to a numerical algorithm developed by Adam Oberman [2005].

Tug of war games with a terminal time

If we impose a terminal time to the tug of war game, we derive the (homogeneous) parabolic equation

$$u_t = \frac{\partial_i u \; \partial_j u}{|\nabla u|^2} \; \partial_{ij} u.$$

Mean curvature flow

The homoegeneous parabolic equation for $p = 1$ reads

$$u_t = \left(\delta_{ij} - \frac{\partial_i u \partial_j u}{|\nabla u|^2}\right) \partial_{ij} u.$$

This is the evolution equation for the function u whose level sets follow a mean curvature flow. This equation was studied by a number of authors like Chen-Giga-Goto, Evans-Spruck, Evans-Soner-Souganidis, Ishii-Souganidis, Oberman, Minicozzi-Colding, etc...

Homogeneous parabolic p-Laplace equation.

Y. Peres and S. Shefield [2008] extended the tug of war game to a construction of the p-Laplace equation by adding lateral noise. When we add a terminal time to this game, we obtain the homogeneous parabolic equation

$$u_t = \triangle u + (p-2)\frac{\partial_i u \partial_j u}{|\nabla u|^2}\partial_{ij} u.$$

This parabolic problem was considered by Manfredi-Parviainen-Rossi [2010]. Existence and uniqueness of Lipschitz viscosity solutions was established by Does [2011] and Banerjee-Garofalo [2013].

Our result

Theorem (Tianling Jin, LS.)

Let u be a viscosity solution of the homogeneous parabolic p-Laplace equation

$$u_t = \triangle u + (p-2)\frac{\partial_i u\, \partial_j u}{|\nabla u|^2}\partial_{ij} u \quad \textit{in } Q_1 = (-1,0] \times B_1,$$

then ∇u *is well defined and Hölder continuous in* $Q_{1/2} = (-1/4, 0] \times B_{1/2}$.

Difficulty: no variational structure. Different methods need to be used.

Uniform ellipticity

We have the equation

$$u_t = \triangle u + (p-2)\frac{\partial_i u\, \partial_j u}{|\nabla u|^2}\partial_{ij} u \qquad \text{in } Q_1 = (-1,0] \times B_1.$$

For

$$a_{ij} = \delta_{ij} + (p-2)\frac{\partial_i u \partial_j u}{|\nabla u|^2}.$$

Note that $\max(p-1,1)\mathrm{I} \geq \{a_{ij}\} \geq \min(p-1,1)\mathrm{I}$. The equation is uniformly elliptic in non-divergence form.

The coefficients $a_{ij}(\nabla u)$ are a smooth function of ∇u except where $\nabla u = 0$. If $a_{ij}(\nabla u)$ was smooth everywhere resp. ∇u, the regularity of the solution would follow from classical estimates.

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Lipschitz estimates

Fact 1. $W^{2,\varepsilon}$ estimates

Solutions to uniformly parabolic equations

$$u_t = a_{ij}(x,t)\partial_{ij}u,$$

with $\lambda \mathrm{I} \leq \{a_{ij}\} \leq \Lambda \mathrm{I}$, are in $W^{2,\varepsilon}$ for some $\varepsilon > 0$. This means that

$$\int_{Q_1} |D^2 u|^\varepsilon \, \mathrm{d}x \leq C \left(\sup_{Q_1} |u| \right)^\varepsilon,$$

for some $\varepsilon > 0$ and C universal.

Fact 2. $|\nabla u|^p$ is a subsolution to a unif. parabolic equation

Fact 3. Local Maximum principle

Lipschitz estimates

Fact 1. $W^{2,\varepsilon}$ estimates

$$\int_{Q_1} |Du|^{\varepsilon} \, \mathrm{d}x \leq C \left(\max_{Q_1} |u| \right)^{\varepsilon} .$$

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Fact 2. $|\nabla u|^p$ is a subsolution to a unif. parabolic equation

The function $\varphi = |\nabla u|^p$ is a subsolution to

$$\varphi_t - a_{ij}\partial_{ij}\varphi \le 0.$$

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Fact 3. Local Maximum principle

Non negative subsolutions to uniformly parabolic equations in non-divergence form

$$\varphi_t - a_{ij}(x,t)\partial_{ij}\varphi \le 0,$$

satisfy the local maximum principle

$$\varphi(0) \le \left(\int_{Q_1} \varphi^{\varepsilon} \,\mathrm{d}x \,\mathrm{d}t \right)^{1/\varepsilon}.$$

Lipschitz estimates

Fact 1. $W^{2,\varepsilon}$ **estimates**

$$\int_{Q_1} |Du|^\varepsilon \, \mathrm{d}x \le C \left(\max_{Q_1} |u| \right)^\varepsilon .$$

Fact 2. $|\nabla u|^p$ **is a subsolution to a unif. parabolic equation**

The function $\varphi = |\nabla u|^p$ is a subsolution to

$$\varphi_t - a_{ij} \partial_{ij} \varphi \le 0.$$

Fact 3. Local Maximum principle

$$\varphi(0) \le \left(\int_{Q_1} \varphi^\varepsilon \, \mathrm{d}x \, \mathrm{d}t \right)^{1/\varepsilon} .$$

$$\implies |\nabla u(0)| \le C \max_{Q_1} |u|.$$

The alternative proof by Does uses Berstein's technique.

The oscillation of ∇u

We aim at proving that

C^α regularity for ∇u

$$\nabla u(Q_{r^k}) \subset B_{(1-\delta)^k}(p_k).$$

for some $r, \delta \in (0,1)$ and some sequence of centers $p_k \in \mathbb{R}^n$ and all $k \geq 0$.

This is exactly the C^α regularity of ∇u where $\alpha = \log(1-\delta)/\log r$.

In order to prove it by induction, we must show

(flawed) induction step

$$\nabla u(Q_1) \subset B_1 \Longrightarrow \nabla u(Q_r) \subset B_{(1-\delta)},$$

for some $r, \delta \in (0,1)$.

Improvement of oscillation

Lemma

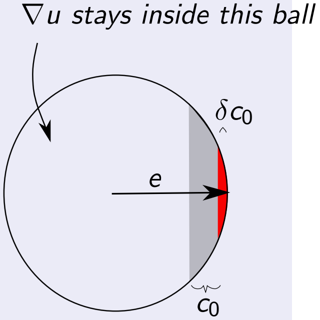
Assume $\nabla u(Q_1) \subset B_1$. Let e be any unit vector. Assume that

$$|\{(t,x) \in Q_1 : e\cdot\nabla u(t,x) \leq 1-c_0\}| \geq \mu.$$

Then

$$e \cdot \nabla u(t,x) \leq 1 - \delta c_0 \quad \text{in } Q_r.$$

Here δ and r are positive and depend on p and μ.



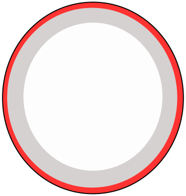
Proof. The function $w = \max(e \cdot \nabla u, 1 - c_0)$ is a subsolution of some parabolic equation.
Indeed, note that the equation is only relevant where $1 - c_0 \leq e \cdot \nabla u \leq 1$. Our equation is smooth if the gradient is restricted there. So we can differentiate the equation and obtain something.

The favorable case

If we can apply the previous lemma for fixed μ and c_0 and for all directions e, then we obtain that

$$\nabla u(Q_r) \subset B_{(1-\delta)},$$

and the induction step succeeds.



∇u never goes into the red area.

This indution step can only work indefinitely if $\nabla u(0) = 0$. We cannot expect this to always happen. We must have a **backup plan** for the case when the conditions of the Lemma are not met. We can choose arbitrarily small μ and c_0.

Small perturbation of smooth parabolic equations

Theorem (Yu Wang [2013])

Let u be a solution to the parabolic equation

$$u_t = F(D^2 u, \nabla u) \text{ in } Q_1.$$

Assume F is smooth and uniformly elliptic in a neighborhood of $(D^2\varphi, \nabla\varphi)$ for some smooth solution φ. **If $\|u - \varphi\|_{L^\infty}$ is sufficiently small, then $u \in C^{2,\alpha}$** *in $Q_{1/2}$.*

This is the parabolic version of an earlier result by Ovidiu Savin for elliptic equations.

The backup plan

It will eventually happen that our previous lemma does not apply. That is, for some unit vector e,

$$|\{(t,x) \in Q_1 : e \cdot \nabla u(t,x) \leq 1 - c_0\}| < \mu. \qquad (1)$$

The constants μ and c_0 are arbitrarily small.

We want to show that in this case $u(x,t) - e \cdot x$ has a small oscillation in $Q_{1/2}$ and we can apply the result of Yu Wang.

The condition (1) tells us that for fixed time $\boldsymbol{u(x,t) - p \cdot x}$ has small oscillation, except for a set of times of small measure.

Small oscillation for all fixed times

The condition (1) tells us that for fixed time $\boldsymbol{u(x,t) - p \cdot x}$ has small oscillation, except for a set of times of small measure.

Recall that the function $v(x,t) = u(x,t) - p \cdot x$ solves a uniformly parabolic equation

$$\boldsymbol{v_t = a_{ij}\partial_{ij} v},$$

with $\lambda \mathrm{I} \leq \{a_{ij}\} \leq \Lambda \mathrm{I}$.

Using the C^α estimates of Krylov and Safonov, we extend the small oscillation for **all values of $\boldsymbol{t}$.**

Small oscillation for the whole parabolic cylinder

Lemma

Let v be a solution to the uniformly parabolic equation $v_t = a_{ij}\partial_{ij} v$. *Assume that*

$$\operatorname*{osc}_{x \in B_1} v(t,x) \leq \delta \text{ for all } t \in [-1,0].$$

Then,

$$\operatorname*{osc}_{Q_1} v(t,x) \leq C\delta,$$

for a constant C depending on dimension and the ellipticity constants.

Proof. Use barriers of the form

$$w(t,x) = a + \delta|x|^2 + C\delta t.$$

This is a supersolution for large enough C.

Summary of strategy

For as long as we can apply the first lemma, we get

$$\nabla u(B_{r^k}) \subset B_{(1-\delta)^k}.$$

Whenever the first lemma fails, that means there is a unit vector e so that ∇u is very close to e at most points in Q_1.

Using the uniform ellipticity of the equation (in non-divergence form) we deduce that $u(t,x) - p \cdot x$ has a small oscillation and we conclude applying the result of Yu Wang.

Both alternative co-exist peacefully and we obtain a uniform C^α estimate for ∇u in $Q_{1/2}$.