Partial regularity for fully nonlinear PDE

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Joint work with Scott Armstrong and Charles Smart

Outline

Introduction
- Intro
- Review of fully nonlinear elliptic PDE
- Our result

Preliminaries
- $W^{3,\varepsilon}$ estimate
- Flat solutions

Proof of partial regularity

Unique continuation
- Intro to unique continuation

Fully nonlinear elliptic PDE

We consider equations of the form

$$F(D^2u) = 0 \qquad \text{in } B_1.$$

with

$$\lambda I \leq \frac{\partial F}{\partial X_{ij}} \leq \Lambda I \qquad \text{(uniform ellipticity)}.$$

The Dirichlet problem in B_1 has a unique viscosity solution, which a priori is just a continuous function.

Basic question: is the viscosity solution going to be C^2?

Regularity results

- The solution is always $C^{1,\alpha}$

  (Krylov and Safonov. Early 80's)

- In 2D, the solution is always $C^{2,\alpha}$

  (Nirenberg 50's)

- In 12D, there are solutions which are not C^2

  (Nadirashvili-Vladut 2007)

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Extra structure conditions

- F concave or convex $\Longrightarrow$ the solution is $C^{2,\alpha}$ (Evans - Krylov, 1983)
- What if F is smooth?
- What if F is homogeneous?
- What if F depends only on the eigenvalues of D^2u?

For any of the three possibilities above, Nadirashvili and Vladut proved that there can be singular solutions in high dimension.

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Partial regularity

Question: Can we show that u is C^2 except in a very small set?

We want to prove an upper bound for the Hausdorff dimension of the possible singular set.

It would be great if we could prove that the singular set of u has Hausdorff dimension at most $n-9$ or even $n-2$. But right now we are far from that.

What we can prove is the following.

Theorem (Armstrong, S., Smart)

If the equation $F(D^2u)=0$ is uniformly elliptic and $F \in C^1$, then u is $C^{2,\alpha}$ outside of a closed set of Hausdorff dimension at most $n-\varepsilon$

(for ε being a small universal constant)

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Reason for the $C^{1,\alpha}$ estimate

The main reason why the $C^{1,\alpha}$ regularity holds is because the derivatives of a solution satisfy a uniformly elliptic equation with *rough* coefficients.

$$F(D^2u) = 0$$

with $\lambda I \leq a_{ij}(x) \leq \Lambda I$.

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$W^{2,\varepsilon}$ estimate

Solutions to uniformly elliptic equations with rough coefficients

$$a_{ij}(x)\partial_{ij}v = 0 \qquad \text{with } \lambda I \leq a_{ij}(x) \leq \Lambda I,$$

also satisfy a $W^{2,\varepsilon}$ estimate which says that v is second differentiable almost everywhere and $D^2 u \in L^{\varepsilon}$.

First proved by Fanghua Lin in 1986. Later, another proof was given by Caffarelli in 1989.

More precise $W^{2,\varepsilon}$ estimate

Solutions to uniformly elliptic equations with rough coefficients

$$a_{ij}(x)\partial_{ij}v = 0 \text{ in } B_1 \qquad \text{with } \lambda I \le a_{ij}(x) \le \Lambda I,$$

satisfy the following estimate. Let

$$A_t := \{x \in B_{1/2} : \text{there exists } L \text{ linear s.t. } |v(y) - L(y)| \le t|x-y|^2 \text{ for all } y \in B_1\}.$$

Then

$$|B_{1/2} \setminus A_t| \le C t^{-\varepsilon} ||v||^{\varepsilon}_{L^\infty(B_1)}.$$

A $W^{3,\varepsilon}$ estimate

The previous $W^{2,\varepsilon}$ estimate can be applied to the derivatives of a solution to a fully nonlinear PDE $F(D^2u) = 0$.
Let

$$A_t := \{x \in B_{1/2} : \text{there exists } P \text{ quadratic s.t.} \\ |u(y) - P(y)| \leq t|x-y|^3 \text{ for all } y \in B_1\}.$$

Then

$$|B_{1/2} \setminus A_t| \leq C t^{-\varepsilon} ||u||^{\varepsilon}_{L^\infty(B_1)}.$$

Almost the same estimate was recently used by Caffarelli and Souganidis to obtain convergence rates for finite difference schemes and homogenization of fully nonlinear PDE.

Flat solutions are $C^{2,\alpha}$

Theorem (O. Savin 2007)

If u solves a uniformly elliptic equation $F(D^2u) = 0$ in B_1, $F \in C^1$, $F(0) = 0$ and $||u||_{L^\infty(B_1)} \leq \delta$, then $u \in C^{2,\alpha}(B_{1/2})$.

Flat solutions are $C^{2,\alpha}$ (scaled)

Theorem (O. Savin 2007)

If u solves a uniformly elliptic equation $F(D^2u) = 0$ in B_r, $F \in C^1$, P is a quadratic polynomial such that $F(D^2P) = 0$ and $||u - P||_{L^\infty(B_r)} \leq \delta r^2$, then $u \in C^{2,\alpha}(B_{r/2})$.

Proof of partial regularity

Let S be the set of points in $B_{1/2}$ where a solution u is not C^2. Let us cover S with a collection of balls $\{B_j\}$ of radius r. We take a subcover if necessary so that $\{3B_j\}$ covers S and they do not overlap (Vitali covering lemma).

$$S \subset \bigcup_{j=1,\ldots,N} 3B_j.$$

In order to estimate the Hausdorff measure of S, we must find an appropriate upper bound for the number of balls.

Proof of partial regularity

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Recall Savin's result

Theorem

If u solves a uniformly elliptic equation $F(D^2u) = 0$ in B_r, $F \in C^1$, P is a quadratic polynomial such that $F(D^2P) = 0$ and $||u - P||_{L^\infty(B_r)} \leq \delta r^2$, then $u \in C^{2,\alpha}(B_{r/2})$.

Thus, for each ball B_j, there is no polynomial P so that $||u - P||_{L^\infty(B_j)} \leq \delta r^2$.

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Let S be the set of points in $B_{1/2}$ where a solution u is not C^2.

$$S \subset \bigcup_{j=1,\dots,N} 3B_j.$$

For each ball B_j, there is no polynomial P so that $||u - P||_{L^\infty(B_j)} \leq \delta r^2$.

Recall the $W^{3,\varepsilon}$ estimate. The set A_t was defined as

$$A_t := \{x \in B_{1/2} : \text{there exists } P \text{ quadratic s.t.} \\ |u(y) - P(y)| \leq t|x-y|^3 \text{ for all } y \in B_1\}.'$$

Thus, no point in B_j can be in A_t for $t = \frac{\delta}{8r}$.

$$\bigcup_{j=1,\dots,N} B_j \subset B_{1/2} \setminus A_t$$

Proof of partial regularity

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$$S \subset \bigcup_{j=1,\dots,N} 3B_j.$$

$$\bigcup_{j=1,\dots,N} B_j \subset B_{1/2} \setminus A_t$$

Recall that the $W^{3,\varepsilon}$ estimate says that

$$|B_{1/2} \setminus A_t| \leq C t^{-\varepsilon} ||u||^{\varepsilon}_{L^\infty(B_1)}.$$

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Recall that the $W^{3,\varepsilon}$ estimate says that

$$|B_{1/2} \setminus A_t| \leq C t^{-\varepsilon} ||u||^{\varepsilon}_{L^\infty(B_1)}.$$

$$N r^n \leq C \left(\frac{\delta}{r}\right)^{-\varepsilon}.$$

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Recall that the $W^{3,\varepsilon}$ estimate says that

$$|B_{1/2} \setminus A_t| \leq Ct^{-\varepsilon}||u||^{\varepsilon}_{L^\infty(B_1)}.$$

$$N \leq Cr^{\varepsilon-N}.$$

Second part of the talk

Unique continuation for fully nonlinear PDE

Joint work with Scott Armstrong

Unique continuation problem

Assume u and v are two solutions to a fully nonlinear elliptic equation

$$F(D^2u) = F(D^2v) = 0 \qquad \text{in } B_1.$$

If $\{u = v\}$ has nonempty interior, is it true that $u \equiv v$ in B_1?

case $F(D^2u) = \Delta u$

The unique continuation property certainly holds for harmonic functions. There are three independent proofs.

1. Analyticity: u and v are analytic, therefore unique continuation holds (prehistoric).
2. Carleman estimates (1930's).
3. Frequency formula (Garofalo - Lin 1987).

The last two methods generalize to elliptic equations with variable coefficients

$$a_{ij}(x)\partial_{ij}u = 0,$$

provided that a_{ij} is uniformly elliptic and Lipschitz.

Trivial case: smooth solutions

If $F \in C^{1,1}$ and u and v are two $C^{2,\alpha}$ classical solutions then the unique continuation property holds.

Proof.

From Schauder estimates, u and v are $C^{3,\alpha}$. The difference $w = u - v$ satisfies the elliptic equation

$$a_{ij}(x)\partial_{ij} w = 0$$

with coefficients given by

$$a_{ij}(x) = \int_0^1 (tD^2u(x) + (1-t)D^2v(x))\,\mathrm{d}t.$$

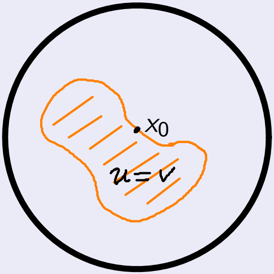
From our smoothness assumptions, $a_{ij}(x)$ are Lipschitz and the linear unique continuation theorems apply. □

Almost trivial case

If $F \in C^{1,1}$ and u and v are two $C^{2,\alpha}$ classical solutions outside of a singular set S with H^{n-1}-measure zero, then the unique continuation property holds.

Proof.

There will be one point on $\partial\{u = v\}$ where both u and v are smooth.



□

Our result

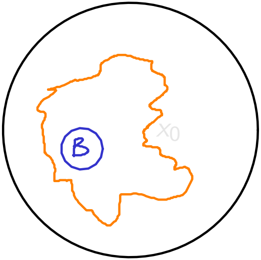
Theorem (Armstrong, S.)

Let u be a viscosity solution to the uniformly elliptic equation $F(D^2u) = 0$ in B_1. Assume that $F \in C^{1,1}$ and $\{u = 0\}$ has nonempty interior. Then $u \equiv 0$ everywhere.

The unique continuation property holds if u is an arbitrary viscosity solution and v smooth.

Proof of the unique continuation result

Let B be a ball contained in $\{u = 0\}$ (it exists by assumption).



We can move B until ∂B and $\partial\{u = 0\}$ have a common point x_0.
We will prove that around this particular point x_0, the solution u is smooth ($C^{2,\alpha}$).

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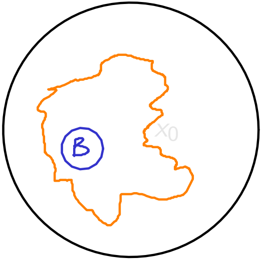

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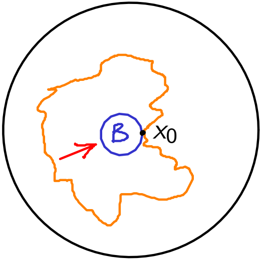

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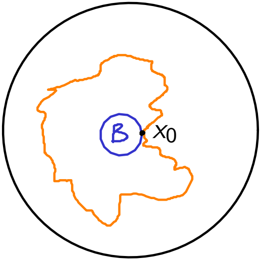

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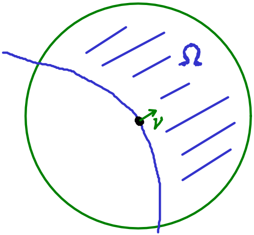
Boundary Harnack for nondivergence equations

Assume that Ω has a smooth boundary (the exterior of a ball for example). Let $x_0 \in \partial\Omega \cap B$. If v is a solution to a uniformly elliptic equation with *rough* coefficients



$$a_{ij}(x)\partial_{ij} v = 0 \quad \text{in } B \cap \Omega,$$

and ν is the unit normal to $\partial\Omega$ at x_0, then there is a an $a \in \mathbb{R}$ such that

$$v(x) = a(x - x_0) \cdot \nu + O(|x - x_0|^{1+\alpha}).$$

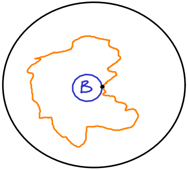
Result obtained by Krylov and independently by Baumann in the early 80's.

There is a simple proof by Caffarelli which is unpublished.

Proof

We apply the boundary Harnack theorem to all derivatives of the solution u to obtain

$$u(x) \leq C|x - x_0|^{2+\alpha}.$$



But then for $r \ll 1$, we will have $||u||_{L^\infty(B_r)} \leq \delta r^2$ and we can apply Savin's result.