Smooth approximations to solutions of nonconvex fully nonlinear elliptic equations

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To Nina Uraltseva with our admiration.

Abstract

We show that fully nonlinear elliptic PDEs (which may not have classical solutions) can be approximated with integro-differential equations which have $C^{2,\alpha}$ solutions. For these approximated equation we prove a uniform $C^{1,\alpha}$ estimate. We also study the rate of convergence.

1 Introduction

The possibility of approximating solutions of nonconvex fully nonlinear equations by classical solutions has been a long standing issue. The recent work [2] provided such approximation but it required considerably technical arguments. The ideas in this article are based in the work [3] on integral fully nonlinear equations

Let φ be a compactly supported, smooth, symmetric, probability density and given a smooth function u, let

$$I_\varepsilon(x) = \int_{\mathbb{R}^n} (u(x+y) + u(x-y) - 2u(x)) \frac{1}{\varepsilon^{n+2}} \varphi\left(\frac{y}{\varepsilon}\right) \, \mathrm{d}y$$

If we let $\varepsilon \to 0$, I_ε converges to an integral on the unit sphere

$$I_0(x) = \int_{S_1} u_{\sigma\sigma}(x) \Phi(\sigma) \, \mathrm{d}\sigma$$

where the weight $\Phi(\sigma)$ is the second moment of φ in the direction σ

$$\Phi(\sigma) = \int_0^\infty t^2 \varphi(t\sigma) \, \mathrm{d}t$$

In turn, $u_{\sigma\sigma} = \sum \sigma_i \sigma_j \partial_{ij} u$ and thus

$$I_0(x) = \sum a_{ij} \partial_{ij} u$$

where the coefficients a_{ij} are

$$a_{ij} = \int_{S_1} \sigma_i \sigma_j \Phi(\sigma) \, \mathrm{d}\sigma$$

conversely, given an elliptic (positive definite) matrix $a_{ij} = B^2 = B^t B$, if φ is a radially symmetric probability density with quadratic moments in any direction equal to $1/|S_1|$ and we define

$$\tilde{\varphi}(x) = \frac{1}{\det B} \varphi(B^{-1}x)$$

then the resulting operator of the limiting process with $\tilde{\varphi}$ is indeed $a_{ij}\partial_{ij}u$.

Consider now the Isaacs fully nonlinear equation

$$\begin{aligned} &F(D^2u) = f(x) \quad \text{in } \Omega \\ &\text{where } F(D^2u) = \inf_\beta \sup_\gamma a_{ij}^{\gamma\beta}\partial_{ij}u = \inf_\beta \sup_\gamma L_{\gamma\beta}u \end{aligned} \tag{1.1}$$

Assume that the equation is uniformly elliptic: there exist constants $0 < \lambda \leq \Lambda$ such that for every index α, β, $\lambda I \leq \{a_{ij}^{\alpha\beta}\} \leq \Lambda I$. Thus, we can write every linear operator as

$$a_{ij}^{\alpha\beta}\partial_{ij}u = \frac{\lambda}{2}\triangle u + \lim_{\varepsilon\to 0} I_\varepsilon^{\alpha\beta}(v)$$

where

$$I_\varepsilon^{\alpha\beta}v = \int_{\mathbb{R}^n} \delta u(x,y)\frac{1}{\varepsilon^{n+2}\det B_{\alpha\beta}}\varphi\left(B_{\alpha\beta}^{-1}\frac{y}{\varepsilon}\right)\,\mathrm{d}y,$$

and $B_{\alpha\beta}^2 = \{a_{ij}^{\alpha\beta}\} - \frac{\lambda}{2}I$. We are using the notation $\delta u(x,y) = u(x+y) + u(x-y) - 2u(x)$, and φ is a compactly supported radially symmetric probability density with second moment equal to one in any direction.

The above discussion suggests that solutions v of the equation

$$E_\varepsilon(v) = \frac{\lambda}{2}\triangle v + \inf_\beta \sup_\alpha I_\varepsilon^{\alpha\beta}(v) = f(x) \tag{1.2}$$

would be, as ε goes to zero, approximations of the solution u to the original fully nonlinear elliptic PDE (1.1). The interesting fact about these approximations is that since the second term is always a Lipschitz function for every $\varepsilon > 0$, then the functions v will be $C^{2,\alpha}$ (with estimates depending on ε). Moreover, since the first term $\frac{\lambda}{2}\triangle v$ is fixed and the second term has some type of ellipticity, it is possible to obtain some a priori estimates independent of ε which coincide, in fact, with known regularity results for fully nonlinear elliptic PDE without convexity assumptions.

In this note we intend to prove existence, smoothness, and rate of convergence to the solution as $\varepsilon \to 0$. We will prove that the problem is well posed for every $\varepsilon > 0$ once boundary value g is extended in a neighborhood of Ω. The we will show that the equation has an interior $C^{1,\alpha}$ estimate independent of ε (corresponding to the interior $C^{1,\alpha}$ estimate for fully nonlinear PDEs). And in the last section we will show that the rate of convergence of the solutions to our approximated problem to the solution to (1.1) is of the order ε^α for some $\alpha > 0$.

2 Some simple properties of the approximate equation

In this section we construct the approximate equation and observe some elementary properties.

To fix ideas, let us choose a smooth function φ that is supported in B_2, $0 \leq \varphi \leq 1$ in its support, and for every direction σ on the unit sphere,

$$\int_0^\infty t^2\varphi(t\sigma)\,\mathrm{d}t = \frac{1}{|S_1|}.$$

This choice is made so that for any C^2 function u,

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n}(u(x+y) + u(x-y) - 2u(x))\frac{1}{\varepsilon^{n+2}}\varphi\left(\frac{y}{\varepsilon}\right)\,\mathrm{d}y = \triangle u(x).$$

And also,

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n}(u(x+y)+u(x-y)-2u(x))\frac{1}{\varepsilon^{n+2}\det(B)}\varphi\left(B^{-1}\frac{y}{\varepsilon}\right)\,\mathrm{d}y = a_{ij}\partial_{ij}u(x),$$

where $B^2=\{a_{ij}\}$.

By construction all matrices $B^{\alpha\beta}$ in the definition of E_ε have their eigenvalues bounded above and below, so the support of $\frac{1}{\det B}\varphi(B^{-1}x)$ will always be contained in a ball B_Q where Q is a universal constant given by $2/\sqrt{\lambda}$. Therefore, the value of $E_\varepsilon u(x)$ depends only on the values of u in $B_{Q\varepsilon}(x)$. This universal constant Q will appear in several estimates in this paper.

Let g be a continuous function defined on $\partial\Omega$. We want to solve the approximated problem $E_\varepsilon u = f$ in Ω and $u=g$ on $\partial\Omega$. However, in order for every $I_\varepsilon^{\alpha\beta}$ to be well defined, we need to extend g to a neighborhood of Ω of size $Q\varepsilon$.

A natural continuous extension of g to $\mathcal{C}\Omega$ can be done by a *sup-convolution*:

$$\tilde g(x) = \sup_{x^*\in\partial\Omega} g(x^*) - |x-x^*|^2$$

For the existence of the solution of the equation for a positive value ε, any continuous extension would work. For computing the rates of convergence, it is better to extend g in a way that the extension is as regular as the original function g. If g is $C^{1,1}$ and Ω has the exterior ball condition, then $\tilde g$ will be $C^{1,1}$ in a neighborhood of $\partial\Omega$. The approximated problem that we will study is

$$\begin{aligned} E_\varepsilon u &= f(x) \quad \text{in } \Omega \\ u &= \tilde g \quad \text{in } (\partial\Omega)_{Q\varepsilon} \end{aligned} \tag{2.1}$$

where $(\partial\Omega)_{Q\varepsilon} = \{x\in\mathcal{C}\Omega : \operatorname{dist}(x,\partial\Omega) < Q\varepsilon\}$ is a $Q\varepsilon$ neighborhood of $\partial\Omega$.

We point out the scaling of the equation. If u is a solution of $E_\varepsilon u = f$, then $\bar u(x) = su(tx)$ is a solution of $\bar E_{t\varepsilon}(u) = st^2 f$ where $\bar E_{t\varepsilon}$ is also an operator as in (1.2) but with $t\varepsilon$ instead of ε.

We note that the equation is constructed so that $E_\varepsilon P = F(D^2P)$ for any quadratic polynomial P, where F is the original fully nonlinear equation. The following lemmas provide more precise statements.

Lemma 2.1. *Assume φ radially symmetric, nonnegative and supported in a ball B_R. If u is superharmonic in $B_R(x)$ then*

$$Iu(x) = \int_{\mathbb{R}^n}\delta u(x,y)\varphi(y)\,\mathrm{d}y \le 0$$

Proof. It is a simple average of the mean value theorem. □

Corollary 2.2. *Let u be a function such that for a fixed matrix a_{ij},*

$$a_{ij}\partial_{ij}u \le 0 \quad \textit{in a ball } B_{r+Q\varepsilon}, \quad (\textit{resp.} \ge 0)$$

then also

$$\int_{B_{Q\varepsilon}}\delta u(x,y)\frac{1}{\det B}\varphi(B^{-1}y)\,\mathrm{d}y \le 0 \quad \textit{in } B_r \quad (\textit{resp.} \ge 0)$$

Proof. We apply Lemma 2.1 to $\tilde u(x) = u(Bx)$. □

Corollary 2.3. *Let λ_{min} and λ_{max} be the smallest and largest eigenvalues of D^2u. Assume that in a ball $B_{Q\varepsilon}$, $\lambda_{min}\le 0\le\lambda_{max}$ and $-\lambda_{min}\ge c(\lambda,\Lambda)\lambda_{max}$, φ as above, then*

$$\int_{B_{Q\varepsilon}}\delta u(x,y)\frac{1}{\det B}\varphi(B^{-1}y)\,\mathrm{d}y \le 0$$

Here B is a positive definite matrix and λ and Λ above refers to the ellipticity constants.

Proof. Under the change of variables $y^* = B^{-1}y$, u remains superharmonic if $C(\lambda, \Lambda)$ is large enough. $\square$

Recalling that E_ε is given by the Laplacian plus an inf-sup of integral operators as in Corollary 2.3, we have the following corollary.

Corollary 2.4. *If λ_{min} and λ_{max} are the smallest and largest eigenvalues of D^2u, $\lambda_{min} \leq 0 \leq \lambda_{max}$ and $-\lambda_{min} \geq c(\lambda, \Lambda)\lambda_{max}$, φ as above, then*

$$E_\varepsilon u \leq 0$$

3 Subsolutions, supersolutions, and Perron's method

In this section we will prove that the approximated problem has a unique solution using Perron's method. The smoothness of the integral part of the equation allows us to use very classical arguments.

We will assume that in the domain Ω is connected and the classical Laplace equation is solvable for any continuous Dirichlet data. We will use this to show that the solution to our approximated problem also achieves its boundary data g continuously. For example we can consider a connected domain Ω with a uniform external ball condition: there exists a ρ_0 such that for every point $x \in \partial\Omega$, there exists a ball $B_{\rho_0}(y)$ contained in $\mathcal{C}\Omega$ such that $x \in \partial B_{\rho_0}(y)$.

We point out that since we are using a φ that is smooth and compactly supported, all functions $I_\varepsilon^{\alpha\beta}u$ are uniformly smooth (for given $\varepsilon > 0$) even if u is only L^1_{loc}. Therefore, if $u \in L^1_{loc}$, the function $\inf_\beta \sup_\alpha I_\varepsilon^{\alpha\beta}u$ will be locally Lipschitz. This allows us to understand the notion of subsolutions and supersolutions in a classical distributional sense. Even for $u \in L^1_{loc}$, we can make sense of

$$\frac{\lambda}{2}\triangle u + \inf_\alpha \sup_\beta I_\varepsilon^{\alpha\beta}u \geq f(x)$$

in the sense of distributions since the second term is just a Lipschitz function.

Lemma 3.1. *If $u \in L^1_{loc}$ is a supersolution (resp. subsolution) of the equation:*

$$\frac{\lambda}{2}\triangle u + \inf_\beta \sup_\alpha I_\varepsilon^{\alpha\beta}u \leq f(x) \quad \textit{(resp. } \geq f(x)\textit{)} \ \ \textit{in } \Omega$$

then u is lower semicontinuous

Proof. From the equation we see that locally $\triangle u$ is bounded above. It is a classical result that this implies that u has a lower semicontinuous representative. $\square$

We now prove the strong comparison principle.

Lemma 3.2 (Comparison principle)**.** *Assume Ω is a connected domain. Let u and v be a subsolution and a supersolution respectively. If $u - v$ has an absolute maximum at some interior point in Ω, then $u - v$ is constant in Ω.*

Proof. Assume $u - v$ assumes a positive maximum at some interior point $x \in \Omega$. Then for any indexes α, β, we would have

$$I_\varepsilon^{\alpha\beta}(u-v)(x) = \frac{1}{\varepsilon^{n+2}\det B_{\alpha\beta}} \int \delta(u-v)(x,y)\varphi\left(B_{\alpha\beta}^{-1}\frac{y}{\varepsilon}\right)\, \mathrm{d}y$$

But we see that since $u-v$ achieves its absolute maximum at x, then $\delta(u-v)(x,y) \leq 0$ for every $y \in \mathbb{R}^n$. Moreover, either $u = v$ in a neighborhood of x or $I_\varepsilon^{\alpha\beta} u \leq -\eta_0$ for any α, β (with η_0 depending on the function $u-v$ and the ellipticity constants λ and Λ).

Assume the latter, then $\inf \sup I_\varepsilon^{\alpha\beta} u \leq -\eta_0$. And this is a Lipschitz function so it must be negative in a neighborhood of x. This means that $\triangle(u-v) > 0$ in a neighborhood of x, which is clearly impossible since $u-v$ has a maximum at x.

Thus we conclude that u must be equal to v in a neighborhood of x. By the classical connectedness argument, $u = v$ in the whole domain. $\square$

The uniqueness of the solution to the equation (2.1) is an immediate consequence of the comparison principle. We will also use it to prove existence of solution by following a more or less classical Perron's method approach.

Let u be the infimum of all supersolutions

$$\begin{aligned} E_\varepsilon v &\leq f \quad \text{in } \Omega \\ v &\geq g \quad \text{in } (\partial\Omega)_{Q\varepsilon} \end{aligned}$$

We will prove that u is the solution to equation (2.1) provided that f and g are continuous. As usual for Perron's method, we divide the proof in showing that u is a solution to the equation inside the domain Ω and showing that u achieves continuously the boundary values on $\partial\Omega$.

First of all, we must make sure that the infimum u is well defined.

Lemma 3.3. *Assume that g and f are bounded continuous functions. The function u above is well defined and bounded.*

Proof. We must find an upper and lower bound.

Let $M = \sup |f|$ and $N = \sup |g|$. Let R be a large enough radius such that the ball B_R contains an ε neighborhood of Ω. Then it is an elementary computation that the functions

$$b^\pm(x) = M(R^2 - |x|^2)^\pm \pm N$$

are respectively a supersolution and a subsolution of equation (2.1) since they are second order polynomials and thus the value of $E_\varepsilon b^\pm$ coincides with $F(D^2 b^\pm)$.

By the comparison principle (Lemma 3.2), every supersolution v is larger than b^- in the whole domain Ω.

On the other hand, for every supersolution v, the function $\min v, b^+$ is also a supersolution. So u is equal to the infimum of all functions $\min v, b^+$, which are all bounded below and above by b^- and b^+. Thus $b^- \leq u \leq b^+$. $\square$

Theorem 3.4. *The function u above satisfies $E_\varepsilon u = f$ in Ω, $u = g$ in $(\partial\Omega)_{Q\varepsilon}$ and it is continuous.*

Proof. We can consider only supersolutions v that are bounded above and below by b^+ and b^- as in the proof of Lemma 3.3. Let M be the maximum value of $|b^\pm|$, so that all v satisfy $|v| \leq M$.

By the boundedness of v and the smoothness of φ, $\inf \sup I_\varepsilon^{\alpha\beta} v$ is uniformly bounded and Lipschitz. In particular, there is a constant C such that for every supersolution v

$$\frac{\lambda}{2} \triangle v \leq -\inf \sup I_\varepsilon^{\alpha\beta} v \leq C.$$

We first prove that u is still a supersolution. It follows from Greens formula that a constant upper bound on the laplacian is equivalent to the inequality

$$v(x) \geq \frac{1}{|B_r|} \int_{B_r(x)} v(y) \, \mathrm{d}y - C\omega_n r^2$$

for any ball $B_r(x) \subset \Omega$, where ω_n is a dimensional constant and C is the same constant as in the inequality above.

This condition is clearly preserved by taking infimum, so $\triangle u$ is also bounded above by the same constant and u is lower semicontinuous. Moreover, let v_k be a sequence of supersolutions such that at a given point x, $v_k(x) \to u(x)$, then for each α, β,

$$\begin{aligned} I_\varepsilon^{\alpha,\beta} v_k(x) &= \frac{1}{\varepsilon^{n+2} \det B_{\alpha\beta}} \int (v_k(x+y) + v_k(x-y) - 2v_k(x)) \varphi\left(B_{\alpha\beta}^{-1} \frac{y}{\varepsilon}\right) \, \mathrm{d}y \\ &\geq \frac{1}{\varepsilon^{n+2} \det B_{\alpha\beta}} \int (u_k(x+y) + u_k(x-y) - 2v_k(x)) \varphi\left(B_{\alpha\beta}^{-1} \frac{y}{\varepsilon}\right) \, \mathrm{d}y \to I_\varepsilon^{\alpha,\beta} u(x) \end{aligned}$$

Let $D = \liminf_{k\to\infty} \inf\sup I_\varepsilon^{\alpha\beta} v_k$. From the above discussion, $D \geq \inf\sup I_\varepsilon^{\alpha\beta} u$. Since all these functions are uniformly Lipschitz and f is continuous, for any $\eta > 0$, we can find a small radius $r_0 > 0$ such that for k large and any $r < r_0$,

$$v_k(x) \geq \frac{1}{|B_r|} \int_{B_r(x)} v_k(y) \, \mathrm{d}y - (2 \frac{-D + f(x)}{\lambda} + \eta) \, \omega_n r^2$$

Again, this condition is preserved by taking infimum, thus $\frac{\lambda}{2} \triangle u \leq -D + \eta$ in $B_{r_0}(x)$. We are proving that for any x in Ω and $\eta > 0$, there is a neighborhood around x such that $\frac{\lambda}{2} \triangle u + \inf\sup I_\varepsilon^{\alpha\beta} u \leq f(x) + \eta$. Therefore, $\frac{\lambda}{2} \triangle u + \inf\sup I_\varepsilon^{\alpha\beta} u \leq f(x)$ in Ω and u is a supersolution.

Now we have to show that u is also a subsolution by proving the opposite inequality. Since u is a supersolution of (2.1), we know that $f - \inf\sup I_\varepsilon^{\alpha\beta} u - \triangle u$ is a nonnegative measure μ. Assume μ is nonzero. So there must be some point x_0 and $\eta > 0$ where $\mu(B_r(x_0)) \geq \eta r^n$ if r is small enough.

Since $\inf\sup I_\varepsilon^{\alpha\beta} u$ is a Lipschitz function and f is continuous, we can find an $r > 0$ such that

$$\sup_{B_r(x_0)} | - \inf\sup I_\varepsilon^{\alpha\beta} u + f + \inf\sup I_\varepsilon^{\alpha\beta} u(x_0) - f(x_0)| < c\eta.$$

for c arbitrarily small.

Let us substitute u in a tiny ball $B_\rho(x)$ by the solution to

$$\frac{\lambda}{2} \triangle \bar{u} = - \inf\sup I_\varepsilon^{\alpha\beta} u(x_0) + f(x_0) - 2c\eta \qquad \text{in } B_\rho(x_0) \tag{3.1}$$

$$\bar{u} = u \qquad \text{on } \partial B_\rho(x_0) \tag{3.2}$$

so that in B_ρ, $\triangle(\bar{u} - u) \geq \mu - 3c\eta$ with $\mu(B_{\rho/2}(x_0)) \geq \eta\rho^n/2^n$. If we choose c small enough (by making ρ small) this implies that $\bar{u} < u$ in $B_\rho(x_0)$.

Since $\bar{u} \leq u$ inside $B_\rho(x_0)$, $\bar{u} = u$ outside $B_\rho(x_0)$, and $E_\varepsilon u \leq 0$, then also $E_\varepsilon \bar{u} \leq 0$ outside $B_\rho(x_0)$.

On the other hand, the value of the difference of the integral terms $|I_\varepsilon^{\alpha\beta} u - I_\varepsilon^{\alpha\beta} \bar{u}|$ can be made arbitrarily small by taking ρ small enough ($\rho \ll \varepsilon$). In particular, it can be made smaller than $c\eta$. Therefore $\bar{u}$ will also be a supersolution of (2.1) inside $B_\rho(x_0)$ if ρ is very small. But then $\bar{u} \geq u$ and this is a contradiction.

The contradiction came from assuming that $f - \inf\sup I_\varepsilon^{\alpha\beta} u - \triangle u$ was non zero. So u must be a solution in Ω.

By construction $u \geq g$ in $(\partial\Omega)_{Q\varepsilon}$. Let us show that $u = g$ in $(\partial\Omega)_{Q\varepsilon}$ and u is continuous on $\partial\Omega$.

Since all v are uniformly bounded, then every integral operator $I_\varepsilon^{\alpha\beta} v$ is uniformly bounded depending for each $\varepsilon > 0$ (depending on ε). That means that we can find a simple supersolution and a subsolution by solving the following problems

$$\begin{aligned} \triangle s^{\pm}(x) &= \mp C \quad \text{in } \Omega \\ s &= g \quad \text{in } (\partial\Omega)_{Q\varepsilon} \end{aligned}$$

where C is the upper bound for all $|I_\varepsilon^{\alpha\beta} v|$. Therefore the functions $\min(s^+, M)$ and $\max(s^-, -M)$ (recall $|v| \leq M$) are a supersolution and a subsolution respectively which have the same value on $(\partial\Omega)_{Q\varepsilon}$ and are continuous on $\partial\Omega$. Therefore u must be in between the two, which implies that $u = g$ in $(\partial\Omega)_{Q\varepsilon}$ and u is continuous on $\partial\Omega$. □

4 Smoothness

We start this section by pointing out that for every $\varepsilon > 0$, the solution u of

$$\begin{aligned} E_\varepsilon u &= f \quad \text{in } \Omega \\ u &= \tilde{g} \quad \text{in } (\partial\Omega)_{Q\varepsilon} \end{aligned}$$

is a C^2 function. Indeed, from construction it is a continuous function. But since φ is smooth and compactly supported, the integral operators $I_{\alpha\beta} u$ are all uniformly Lipschitz. Therefore the term $\inf \sup I_{\alpha\beta} u$ is Lipschitz. Thus $\triangle u$ is Lipschitz which implies that $C^{2,\alpha}$ for every $\alpha < 1$ by the classical estimates for the Laplace equation.

The estimate above on the C^2 norm of u depends on the value of ε. In this section we will prove that an interior $C^{1,\alpha}$ estimate can be obtained independently of ε. The fact that $u \in C^2$ for every $\varepsilon > 0$ means that we are only dealing with classical solutions.

We will obtain a $C^{1,\alpha}$ estimate for u by applying the following proposition to incremental quotients.

Lemma 4.1. *Let $0 < \lambda \leq \Lambda$ and $\varepsilon > 0$. Let u be a solution to the following equation*

$$\frac{\lambda}{2}\triangle u + \int_{\mathbb{R}^n} \delta u(x,y) \frac{1}{\varepsilon^{n+2}\det B(x)} \varphi\left(B(x)^{-1}\frac{y}{\varepsilon}\right) \, \mathrm{d}y = f(x) \quad \text{in } B_1$$

where $B : B_1 \to \mathbb{R}^{n\times n}$ is a matrix valued function such that for every x, $\sqrt{\lambda} I \leq B(x) \leq \sqrt{\Lambda} I$ and f is a bounded function. Then u satisfies the estimate

$$||u||_{C^\alpha(B_{1/2})} \leq C(||u||_{L^\infty(B_{1+Q\varepsilon})} + ||f||_{L^\infty(\Omega)})$$

where C depends on λ, Λ and n, but not on ε or on any modulus of continuity of B.

Instead of proving Lemma 4.1, we will prove a more general result. Because of the bounds from above and below for the eigenvalues of $B(x)$, for every x the function $\varphi\left(B(x)^{-1}\frac{y}{\varepsilon}\right)$ (as a function of y) is nonnegative and supported in some ball $B_{Q\varepsilon}$. So Lemma 4.1 is a particular case of the following lemma.

Theorem 4.2. *Let $Q > 0$ and $\varepsilon > 0$. Let u be a (classical) solution to the following equation*

$$L_\varepsilon u := \triangle u + \int_{B_{Q\varepsilon}} \delta u(x,y) k(x,y) \, \mathrm{d}y = f(x) \quad \text{in } B_1 \tag{4.1}$$

where $k(x,y)$ is a nonnegative function such that $k(x,y) \leq \varepsilon^{-n-2}$ if $|y| < Q\varepsilon$ and f is a bounded function. Then u satisfies the estimate

$$||u||_{C^\alpha(B_{1/2})} \leq C(||u||_{L^\infty(1+Q\varepsilon)} + ||f||_{L^\infty(\Omega)})$$

where C depends on Q and n, but not on ε or on any modulus of continuity of k.

The proof of Lemma 2 uses the classical idea of showing that the oscillation in diadic balls decreases geometrically. For that we will show a growth lemma, whose proof depends on the scale even though the estimate is uniform in scale at the end.

We recall the scaling of the equation. If u is a solution of $L_\varepsilon u = f$ for some operator L_ε as in Theorem 4.2, then $\bar{u}(x) = su(tx)$ is a solution of $\bar{L}_{t\varepsilon}(u) = st^2 f$ where $\bar{L}_{t\varepsilon}$ is also an operator as in Theorem 4.2 but with $t\varepsilon$ instead of ε.

There are two different scales in this problem. When looking at a scale larger then ε, the ellipticity of the integral part of L plays a role. When looking at a finer scale than ε, then the integral term in the equation can be considered just a smooth perturbation for the Laplace equation.

If we want to prove a Hölder continuity result, we must be able to show that the oscillation of the function decreases at all scales. When looking at a fine scale, we must consider rescalings of the original function of the form $\rho^\alpha u(x/\rho)$, which will solve an equation for an operator $L_{\varepsilon/\rho}$ with ε/ρ large if ρ is smaller then ε.

In the next few Lemmas, we write e instead of ε to stress that we will apply the lemmas at different scales. If we apply it at scale ρ, we would need to consider an operator $L_{\varepsilon/\rho}$ as above for which $e = \varepsilon/\rho$ may be large.

Lemma 4.3. *Assume $e > e_0$ (for a large e_0). There exists an $\eta_0 > 0$, $0 < \mu < 1$ and $M > 1$ depending only on Q and n such that if*

$$\triangle u + \int_{B_{Qe}} \delta u(x,y) k(x,y) \, \mathrm{d}y \leq \eta_0 \quad \textit{in } B_1$$
$$u \geq 0 \quad \textit{in } B_{1+Qe}$$
$$\inf_{B_{1/2}} u \leq 1$$

then $|\{u > M\} \cap B_{1/4}| \geq \mu$.

Proof. Let $v = \min(u/M, 1)$. Since $e > e_0$, we can control the L^∞ norm of the integral term.

$$\int_{B_{Qe}} \delta u(x,y) k(x,y) \, \mathrm{d}y \geq -\int k(x,y) \, \mathrm{d}y \geq -Ce_0^{-2} \geq -\eta_0 \ \text{ if } e_0 \text{ is large enough}$$

Thus, we obtain an estimate for the plain Laplacian of the function

$$\triangle v \leq 2\eta_0 \ \text{ in } B_1$$

On the other hand, since $\inf_{B_{1/2}} u \leq 1$, then $\inf_{b_{1/2}} v \leq 1/M$. Therefore the set $\{v \geq M\}$ cannot cover a large portion of $\cap B_{1/4}$ if η_0 is small, i.e.

$$|\{v < M\} \cap B_{1/4}| \geq \mu$$

□

Lemma 4.4. *Assume $e < e_0$, where e_0 is the one from Lemma 4.3. There exists an $\eta_0 > 0$, $0 < \mu < 1$ and $M > 1$ depending only on Q and n such that if*

$$\triangle u + \int_{B_{Qe}} \delta u(x,y) k(x,y) \, \mathrm{d}y \leq f(x) \quad \textit{in } B_1$$
$$u \geq 0 \quad \textit{in } B_{1+Qe}$$
$$\inf_{B_{1/2}} u \leq 1 \quad \textit{and}$$
$$||f||_{L^n(B_1)} \leq \eta_0$$

then $|\{u < M\} \cap B_{1/4}| \geq \mu$.

In order to prove the lemma above, we prove the following version of Alexandroff-Backelman-Pucci estimate at a coarse scale.

Lemma 4.5 (coarse ABP estimate)**.** *Assume $e \leq e_0$. Let $u : B_1 \to \mathbb{R}$ be a function such that $u \geq 0$ in $(\partial B_1)_{Qe_0} = B_{1+Qe_0} \setminus B_1$ and*

$$\triangle u + \int_{B_{eR}} \delta u(x,y) k(x,y)\, \mathrm{d}y \leq f(x) \quad \text{in } B_1$$

for some nonnegative function f with the same assumption in k as in Theorem 4.2.

Let us extend u as zero outside B_1 and let Γ be the convex envelope of u in B_{1+Qe_0}. Then the classical ABP estimate holds:

$$-\min_{B_1} u \leq C \left(\int_{\{u=\Gamma\}} f(x)^n \, \mathrm{d}x \right)^{1/n}$$

Proof. As in the classical proof of the ABP estimate

$$-\min_{B_1} u \leq C |\nabla\Gamma(\{u=\Gamma\})|^{1/n} = C \left(\int_{\{u=\Gamma\}} \det(D^2\Gamma)^n \, \mathrm{d}x \right)^{1/n}$$

For every point $x \in \{u = \Gamma\}$, the integral term in the equation is nonnegative

$$\int_{B_{eR}} \delta u(x,y) k(x,y)\, \mathrm{d}y \geq 0$$

since all incremental quotients are nonnegative if $u(x) = \Gamma(x)$ (because $e < e_0$). Therefore we have $\triangle\Gamma(x) \leq \triangle u(x) \leq f(x)$.

On the other hand, since $D^2\Gamma(x)$ cannot have a negative eigenvalue, then by the arithmetic-geometric mean inequality $\triangle\Gamma(x)/n \geq \det(D^2\Gamma)^{1/n}$. Thus

$$-\min_{B_1} u \leq C \left(\int_{\{u=\Gamma\}} \det(D^2\Gamma)^n \, \mathrm{d}x \right)^{1/n} \leq C \left(\int_{\{u=\Gamma\}} f(x)^n \, \mathrm{d}x \right)^{1/n}$$

□

Proof of Lemma 4.4. We observe that for large p, a smooth function $b(x)$ given by $(|x|^{-p} - 1)^+$ outside of $B_{1/8}$ and some smooth extension inside $B_{1/8}$ is a subsolution $Lb \geq 0$ outside $B_{1/4}$, and Lb is bounded independently of e ($e > e_0$) inside $B_{1/4}$.

Now we apply ABP to $u - b$ and we proceed as in the proof in [1], chapter 4 (Lemma 4.5). □

By combining Lemmas 4.4 and 4.3, we have that the pointwise estimate holds at every scale. We have the corollary that holds for any value of e.

Corollary 4.6. *There exists an $\eta_0 > 0$, $0 < \mu < 1$ and $M > 1$ depending only on Q and n such that for any $e > 0$, if*

$$\triangle u + \int_{B_{Qe}} \delta u(x,y) k(x,y)\, \mathrm{d}y \leq f(x) \quad \text{in } B_1$$

$$u \geq 0 \quad \text{in } B_{1+Qe}$$

$$\inf_{B_{1/2}} u \leq 1 \quad \text{and}$$

$$||f||_{L^\infty(B_1)} \leq \eta_0$$

then $|\{u < M\} \cap B_{1/4}| \geq \mu$.

Proof. We apply either Lemma 4.4 or Lemma 4.3 depending on whether $e \geq e_0$ or $e < e_0$. □

The previous result implies the L^δ estimate.

Corollary 4.7. *There exists an $\eta_0 > 0$, $\delta > 0$ and C depending only on Q and n such that for any $e > 0$, if*

$$\triangle u + \int_{B_{Qe}} \delta u(x,y) k(x,y) \, \mathrm{d}y \leq f(x) \quad \text{in } B_1 \quad \text{and}$$

$$u \geq 0 \quad \text{in } B_{1+Q\varepsilon}$$

$$||f||_{L^\infty(B_1)} \leq \eta_0$$

then

$$|\{u < t\} \cup B_{1/4}| \leq C t^{-\delta} \inf_{B_{1/2}} u$$

for some constant C depending only on Q and n.

Proof. We follow the proof in [1] chapter 4. The L^δ estimate is proved using only an estimate like Corollary 4.6 at every scale. □

We are now ready to prove Theorem 4.2.

Proof of Theorem 4.2. First of all we point out that we can rescale the equation to make $||f||_{L^\infty}$ as small as we wish, and this estimate will be preserved by the C^α scaling thought the proof.

We will prove a decay in the oscillation of balls around the origin.

$$\operatorname*{osc}_{B^{4^{-k}}(0)} u \leq C(1-\theta)^k ||u||_{L^\infty(B_{1/2+Q\varepsilon})} \tag{4.2}$$

for a universal $\theta > 0$, which immediately implies the result with $\alpha = -\log(1-\theta)/\log 4$.

We prove (4.2) by induction. For $k = 0$ it is true with $C = 1$.

Assume if is true for some $k \in \mathbb{N}$ with $C = 1$. We consider two cases, either $4^k Q\varepsilon < 1/8$ or $4^k Q\varepsilon \geq 1/2$.

Let us first discuss the case $4^k Q\varepsilon < 1/2$.

We use a classical idea of De Giorgi. The values of u remain in an interval $[a,b]$ for $x \in B_{2^{-k}}$ with $b - a \leq (1-\theta)^k ||u||_{L^\infty(B_{1/2+Q\varepsilon})}$. For every $x \in B_{4^{-k-1}}$, u is either above or below $(a+b)/2$. So in at least half of the points (in measure), u will have in one side of $(a+b)/2$. Without loss of generality, let us say that it stays above in at least half of the ball:

$$\{u \geq (a+b)/2\} \cap B_{4^{-k-1}}| \geq \frac{1}{2}|B_{4^{-k-1}}|$$

So, we rescale by considering

$$v = \frac{2}{a+b}(u(2^{-2k-1}x) - a)$$

so that v solves an equation like (4.1) but with $2^{2k+1}\varepsilon$ instead of ε and $v \geq 0$ in B_2. In this case $2^{2k+1}Q\varepsilon < 1$, so $B_{1+2^{2k+1}Q\varepsilon} \supset B_2$. Then we can apply corollary 4.7 and obtain

$$\inf_{B_{1/2}} v \geq c|\{v \geq 1\} \cap B_{1/4}| \geq c$$

for some universal constant c. Scaling back to u, this means that $u \geq a + c\frac{a+b}{2}$ in $B_{4^{k+1}}$, so we the inductive step is proved with $\theta = c/2$ and $C = 1$.

The previous iteration will continue for as long as $4^k Q\varepsilon < 1/2$. Let k be the smallest integer such that $4^k Q\varepsilon \geq 1/2$. The previous iteration process will reach k, so that we have

$$\operatorname*{osc}_{B^{4^{-k}}(0)} u \leq (1-\theta)^k ||u||_{L^\infty(B_{1/2+Q\varepsilon})}.$$

Let $v = (1-\theta)^{-k} u(4^{-k}x)$. So that v satisfies

$$\begin{aligned} \operatorname*{osc}_{B_1} v &\leq 1 \\ \operatorname*{osc}_{B_4} v &\leq (1-\theta)^{-1} < 2 \\ \triangle v + \int_{B_1} \delta v(x,y)\tilde{k}(x,y)\,\mathrm{d}y \quad \text{in } B_1 &= 0 \end{aligned}$$

where $\tilde{k}(x,y) = k(4^k x, 4^k y) \leq Q^{-n-2}$ if $|y| \leq 1$ and zero otherwise. But then the integral term in the equation is bounded by a universal constant C (depending only on Q and n, recall that θ is also universal). So $|\triangle v| \leq C$ in B_1.

Therefore, by the C^α estimates of the Laplace equation, there is a universal constant C such that

$$\operatorname*{osc}_{B_r} v \leq Cr^\alpha.$$

Scaling back, (4.2) holds for some universal constant C for all positive values of k. $\square$

We now state the $C^{1,\alpha}$ estimate.

Theorem 4.8. *Let u be a solution of*

$$\begin{aligned} E_\varepsilon u &= f \quad \textit{in } B_1 \\ u &= g \quad \textit{in } B_{1+Q\varepsilon} \setminus B_1 \end{aligned}$$

where g is a bounded function. Then u satisfies the estimate:

$$||u||_{C^{1,\alpha}(B_{1/2})} \leq C\left(\|g\|_{L^\infty} + ||f||_{L^\infty}\right)$$

where α and C are universal constants (they depend on λ, Λ and n, but not on ε).

Note that the above result can be scaled to obtain that if $E_\varepsilon u = f$ in B_r then

$$||u||_{C^{1,\alpha}(B_{r/2})} \leq C\left(\frac{1}{r^{1+\alpha}}\,\|u\|_{L^\infty(B_{r+Q\varepsilon})} + r^{1-\alpha}||f||_{L^\infty(B_r)}\right)$$

Proof. Theorem 4.8 is a standard consequence of Lemma 4.1. The main point of the proof is that if $E_\varepsilon u_1 = f_1$ and $E_\varepsilon u_2 = f_2$ then $L_\varepsilon(u_1 - u_2) = f_1 - f_2$ for some operator L_ε as in Lemma 4.1. Thus we can apply Lemma 4.1 to incremental quotients of u. First we apply Lemma 4.1 to obtain an estimate for u in C^α. Then we can iteratively obtain estimates with higher exponents by applying Lemma 4.1 to incremental quotients of the form $v(x) = (u(x+he) - u(x))/h^\beta$ for any unit vector e and $h > 0$. In this way we can pass for an estimate in C^β to an estimate in $C^{\beta+\alpha}$ as long as $\beta + \alpha \leq 1$. Thus, after a finite number of steps we obtain an estimate of the Lipschitz norm of u, and finally we apply 4.1 to all directional derivatives u_e and finish the proof.

The details of this (by now standard) procedure can be found in [1] (Corollary 5.7). $\square$

5 A Lipschitz estimate almost up to the boundary

In this section we obtain a uniform Lipschitz estimate in the points inside Ω whose distance to the boundary is at least of order ε. This would become an *up to the boundary* regularity estimate as $\varepsilon \to 0$.

In order to obtain this estimate, we construct barriers to be used in domains with the exterior ball condition. In order to get regularity estimates almost up to the boundary that are uniform in ε, we would need to construct barriers that work for every ε (small enough). This is the purpose of this section.

We recall that Ω has the uniform external ball condition if there exists a ρ_0 such that for every point $x \in \partial\Omega$, there exists a ball $B_{\rho_0}(y)$ contained in $\mathcal{C}\Omega$ such that $x \in \partial B_{\rho_0}(y)$.

If Ω has the external ball condition for a radius $\rho_0 > 0$, then when we consider the ε neighborhood $(\partial\Omega)_{Q\varepsilon}$, it also has the exterior ball condition if ε is small, since the exterior boundary of $(\partial\Omega)_{Q\varepsilon}$ is the parallel surface of $\partial\Omega$ at distance $Q\varepsilon$ which has the exterior ball condition with radius $\rho_0 - Q\varepsilon$.

We apply Corollary 2.4 to the function $v = -|x|^{-p}$ with p a large universal constant. If $|x| > Q\varepsilon$ we obtain that $E_\varepsilon v(x) \geq 0$. We will use this fact to create a barrier of the form $v(x) = a - b|x - x_0|^{-p}$ outside of a ball $B_\rho(x_0)$ which touches $(\partial\Omega)_{Q\varepsilon}$ from the exterior. Naturally this is possible assuming that $\rho \leq \rho_0 - Q\varepsilon$ and $\rho > Q\varepsilon$. So, let us say that $\rho = \rho_0/2$ and ε is small enough. Adding an extra quadratic term, we can also make barriers with a nonzero right hand side:

$$E_\varepsilon \left[a - b|x - x_0|^{-p} - \frac{c}{\lambda}|x|^2 \right] \leq -c \quad \text{outside } B_\rho(x_0)$$

We apply this barrier function to prove the following Lemma.

Lemma 5.1. *Let u be a solution to* (2.1)*. Assume Ω has a uniform external ball condition and $g \in C^{1,1}$.*

There is a small universal ε_0 such that if $\varepsilon < \varepsilon_0$ and $x, y \in \Omega$ be such that $\operatorname{dist}(x, \partial\Omega) \leq 2d$ *and* $\operatorname{dist}(x, y) \leq d$*, then $|u(x) - u(y)| \leq C(d + \varepsilon)$ for a universal constant C.*

Proof. Let z_0 be the closest point to x on the exterior boundary of $(\partial\Omega)_{Q\varepsilon}$: $z_0 \in \mathcal{C}\Omega$ and $\operatorname{dist}(x, z_0) \leq d + \varepsilon$. Since Ω has the exterior ball condition (and thus also does the exterior boundary of $(\partial\Omega)_{Q\varepsilon}$) there is a ball $B_\rho(x_0)$ tangent to $\partial(\partial\Omega)_{Q\varepsilon}$ from the outside. The functions

$$A(x) = g(z_0) + b\rho^{-p} - b|x - x_0|^{-p} - \frac{c}{\lambda}|x|^2$$
$$B(x) = g(z_0) - b\rho^{-p} + b|x - x_0|^{-p} + \frac{c}{\lambda}|x|^2$$

satisfy $E_\varepsilon A \leq -c$ and $E_\varepsilon B \geq c$ in Ω. So if we choose $c = \max|f|$ and b depending on the $C^{1,1}$ norm of g, then A will be a supersolution and B a subsolution to the problem (2.1). Thus $B \leq u \leq A$. But the oscillation of A and B as well as $|A - B|$ in $B_d(x)$ is bounded by $C(d + \varepsilon)$ where C is a constant depending on λ, Λ, n and ρ. Therefore $|u(x) - u(y)| \leq C(d + \varepsilon)$. □

Theorem 5.2. *Let u be a solution to* (2.1)*. Assume Ω has a uniform external ball condition and $g \in C^{1,1}$.*

There is a small universal ε_0 such that if $\varepsilon < \varepsilon_0$ and $x \in \Omega$ such that $\operatorname{dist}(x, \Omega) \geq Q\varepsilon$*, then $|\nabla u| \leq C$ for a constant C depending on λ, Λ, n and ρ_0.*

Proof. Let $d = \operatorname{dist}(x, \partial\Omega)/2$. From the assumptions, we know that $Q\varepsilon < d < diam(\Omega)$. From

Lemma 5.1, $\operatorname{osc}_{B_d(x)} u \leq Cd$. Let us consider the function $\bar{u}(z) = \frac{1}{d}u(x+dz)$. Then $\bar{u}$ satisfies

$$E_{\varepsilon/d}\bar{u}(z) = \frac{\lambda}{2}\triangle \bar{u} + \inf_\beta \sup_\alpha I^{\alpha\beta}_{\varepsilon/d}(\bar{u}) = df(x+dz)$$
$$\operatorname*{osc}_{B_{2Q}} \bar{u} \leq C$$

We can apply Theorem 4.8 and from the interior estimate on the gradient conclude that $|\nabla \bar{u}(0)| \leq C$ for some constant C depending on λ, Λ, n and Ω but not on ε. But that implies that $|\nabla u(x)| \leq C$, which finishes the proof. □

6 Rate of convergence

In this section we prove that the solution u_ε to the approximate problem (2.1) approaches the solution u to the original equation (1.1) uniformly with a rate of the form $C\varepsilon^\alpha$ for some small $\alpha > 0$. We state this in the following theorem.

Theorem 6.1. *Assume $g \in C^{1,1}$ and f is a Lipschitz function. There exists a universal constant C and $\alpha > 0$ (depending only on λ, Λ, n and the exterior ball condition ρ_0 of the domain) such that*

$$||u_\varepsilon - u||_{L^\infty} \leq C\varepsilon^\alpha(||g||_{C^{1,1}} + ||f||_{Lip}||$$

This result can be proved as an application of a general result from [2]. We start by recalling the notion of δ-solutions.

Definition 6.2. *Fix $\delta > 0$. We say that a continuous function v is a δ-supersolution (resp. δ-subsolution) of (1.1) in Ω if, for all $x_0 \in \Omega$ such that $B_\delta(x_0) \subset \Omega$, a polynomial P such that $|P| \leq C\delta^{-\sigma}$, for some universal C, $\sigma > 0$, and $P \leq v$ (resp. $P \geq v$) in $B_\delta(x_0)$ can touch v from below (resp. above) at x_0, i.e., $P(x0) = v(x_0)$, only if $F(D^2P) \leq 0$ (resp. $F(D^2P) \geq 0$). Finally, a continuous function v is a δ-solution if it is both δ-supersolution and δ-subsolution.*

This definition is relevant since the solution to our approximated equation (2.1) is a $Q\varepsilon$-solution. We prove that in the following lemma.

Lemma 6.3. *If u solves (2.1), then u is a $Q\varepsilon$-solution of (1.1).*

Proof. If a quadratic polynomial P touches u from above at a point x then on one hand $\triangle P \geq \triangle u$. On the other hand, if $P \geq u$ in $B_{Q\varepsilon}(x)$ then $P \geq u$ in the full domain of integration of every integral, so $I^{\alpha\beta}_\varepsilon P \geq I^{\alpha\beta}_\varepsilon u$ for every α, β. Therefore $E_\varepsilon P \geq E_\varepsilon u$.

Since P is simply a quadratic polynomial, the value of $E_\varepsilon P$ coincides with the value of the original second order elliptic operator $\inf \sup a^{\alpha\beta}_{ij}\partial_{ij}P$. Thus u is a $Q\varepsilon$-solution. □

The following theorem is proved in [2].

Theorem 6.4. *Let Ω be an open subset of $\mathbb{R}^n$ with regular boundary and consider a solution $u \in C^{0,1}(\overline{\Omega})$ of (1.1). Assume that $v^+ \in C^\gamma(\overline{\Omega})$ (resp. $v^- \in C^\gamma(\overline{\Omega})$) is a δ-subsolution (resp. δ-supersolution) of (1.1) for some fixed $\gamma \in (0,1)$. If $v^+ \geq u + c\delta^{\bar{\alpha}}$ (resp. $v^- \leq u - c\delta^{\bar{\alpha}}$) on $\partial\Omega$ for some positive constants c and α, then there exist uniform constants $C > 0$ and $\alpha \in (0,\bar{\alpha})$ such that, for δ sufficiently small,*

$$v^+ \leq u + C\delta^\alpha \quad (\textit{resp. } v^- \geq u - C\delta^\alpha)$$

Combining theorem 6.4 with Theorem 5.2, we can prove Theorem 6.1. Nevertheless, for completeness, we will provide a detailed sketch of the proof of Theorem 6.1 using the ideas from [2]. The proof uses somewhat sophisticated regularity results for fully nonlinear elliptic equations which can be found in [1]. Since we do not aim at the level of generality as in Theorem 6.4 but only to the particular case of the approximated solutions of this paper, we are able to simplify a few steps in the proof.

Proof of Theorem 6.1. By multiplying f, g and u by an appropriate constant, we can assume that $||g||_{C^{1,1}} = ||f||_{Lip} = 1$. We have already shown that the approximation u_ε is uniformly Lipschitz $Q\varepsilon$ away from $\partial\Omega$.

The solution u separates from the boundary value g linearly from the boundary (depending on the exterior ball condition). From Lemma 5.1, the approximation u_ε separates from the boundary value less than $C\varepsilon$ in a $Q\varepsilon$ neighborhood of $\partial\Omega$. So $|u_\varepsilon - u| \leq C\varepsilon$ for all points $x \in \Omega$ such that $dist(x, \partial\Omega) \leq Q\varepsilon$.

Since u is a solution of the fully nonlinear uniformly elliptic equation (1.1), all first derivatives $u_i = \partial_i u$ are in the class of solutions to equations with *measurable coefficients*. More precisely, for every index i, we have $M^+(D^2 u_i) \geq 0$ and $M^-(D^2 u_i) \leq 0$ (where M^+ and M^- are the extremal Pucci operators). There is a result saying that the Hessians of functions in such class are in L^θ for some small $\theta > 0$ (See [4] or [1], Proposition 7.4). In other words $u_i \in W^{2,\theta}(B_1)$, more precisely, for every $t > 0$, every derivative u_i has a paraboloid of opening t tangent from above (or below) except in a set of measure $t^{-\varepsilon}$.

In terms of the value of the function u itself, this means that for every $t > 0$, except in a singular set of measure $t^{-\varepsilon}$, the function u has a second order Taylor expansion meaning that for some second order polynomial P_x (depending on the point x) such that $||P||_{C^{1,1}} \leq t$,

$$|u(x+y) - P_x(y)| \leq Ct|y|^3.$$

In this regular set, that we will call R, we can obtain an estimate of $E_\varepsilon u$. Recall that $E_\varepsilon P = F(D^2 P) = 0$. So the error comes from the integral term in E_ε applied to the cubic part, which is of order $Ct\varepsilon$. Thus $|E_\varepsilon u(x)| \leq Ct\varepsilon$ except in a set of measure $t^{-\theta}$.

Let us choose $t = \varepsilon^{\alpha-1}$ for some $\alpha \in (0,1)$ to be determined later. So we have $|E_\varepsilon u(x) - f(x)| \leq C\varepsilon^\alpha$ except in a set of measure $\varepsilon^{\theta(1-\alpha)}$.

So we have that $|E_\varepsilon u(x)|$ is small except in a set of small measure. The question is how to fill that gap. We will use a sup-convolution to regularize the solution u and from the regularity estimates in u we will estimate its difference with the sup-convolution u^*.

Let u^* be the sup-convolution of u:

$$u^*(x) = \max_{y \in \overline{B_1}} u(y) - \varepsilon^{-\alpha}|x-y|^2.$$

Since u is a Lipschitz function, for every $x \in \Omega$, the maximum in the sup-convolution is achieved at some $y_* \in \overline{\Omega}$, i.e. $u^*(x) = u(y_*) - \varepsilon^{-\alpha}|x - y_*|^2$, for which $|x - y_*| < C\varepsilon^\alpha$. Moreover, $u_* \leq u + C\varepsilon^\alpha$. On the other hand, it is clear by definition that $u^* \geq u$ since $y = x$ is a candidate for the maximum.

Let $M \subset \Omega$ be the set of all points $y \in \Omega$ such that for some $x \in \Omega$, $u^*(x) = u(y) - \varepsilon^{-\alpha}|x-y|^2$. From this definition, at every such point y the graph of u has a tangent paraboloid from above with opening $\varepsilon^{-\alpha}$. Since u is a solution to the uniformly elliptic equation 1.1, Harnack inequality implies that it also has a tangent paraboloid from below with opening $-C\varepsilon^{-\alpha}$ for a universal constant C. Therefore u is differentiable at every point $y \in M$, and ∇u is Lipschitz in M with Lipschitz constant $C\varepsilon^{-\alpha}$. From the fact that y is the point where the maximum in the definition is achieved, the gradient must satisfy $\nabla u(y) = -2\varepsilon^{-\alpha}(y - x)$. Therefore the map $X(y) := x$ is well defined and Lipschitz, with Lipschitz constant bounded above by a universal C.

We will estimate the minimum of the function $v = C\varepsilon^{-\alpha} + u_\varepsilon - u^*$. By choosing C appropriately we can make sure that $v \geq 0$ in a $Q\varepsilon$ neighborhood of ∂Q. For the interior, we will use the ABP estimate from Lemma 4.5.

For every $x \in \Omega$, $x = X(y)$ for some $y \in M$, and $E_\varepsilon u_*(x) \geq E_\varepsilon u(y)$ in the sense that there is a translation of the graph of u around the point y which is tangent from below to the graph of u_* at the point x.

We will estimate $E_\varepsilon u_*(x)$ depending on whether x is the image by $X(y)$ of a point y in the regular set R or not.

If x is the image $X(y)$ of some point $y \in R$, then $E_\varepsilon u_*(x) \geq E_\varepsilon u(y) \geq f(y) - C\varepsilon^\alpha \geq f(x) - C\varepsilon^\alpha$ (using that f is Lipschitz).

If x is any generic point in Ω (not necessarily the image by $X(y)$ of a regular point), then just by the definition of the sup-convolution, u_* has a tangent paraboloid from below with opening $C\varepsilon^{-\alpha}$ and thus $E_\varepsilon u_*(x) \geq -C\varepsilon^{-\alpha}$.

Therefore, the sup-convolution u_* satisfies the following equation in Ω,

$$E_\varepsilon u_* \geq \begin{cases} f(x) - C\varepsilon^\alpha & \text{in } R \\ -C\varepsilon^{-\alpha} & \text{outside } R \end{cases}$$

Therefore the function v is a subsolution of the linearized equation

$$\frac{\lambda}{2}\triangle v + \int_{\mathbb{R}^n} \delta v(x,y) \frac{1}{\varepsilon^{n+2}\det B(x)} \varphi\left(B(x)^{-1}\frac{y}{\varepsilon}\right) \,\mathrm{d}y \leq \begin{cases} C\varepsilon^\alpha & \text{in } R \\ C\varepsilon^{-\alpha} & \text{outside } R \end{cases}$$

where the matrix $B(x)$ satisfies point-wise the ellipticity estimates $\sqrt{\lambda} I \leq B(x) \leq \sqrt{\Lambda} I$ but may be discontinuous respect to x.

We apply Lemma 4.5 in the set $\Omega_0 = \{x \in \Omega : dist(x, \partial\Omega) \geq Q\varepsilon\}$ and obtain

$$\min -v \leq C\left(\int_R \varepsilon^{\alpha n} \,\mathrm{d}x + \int_{\Omega_0 - R} \varepsilon^{-\alpha n} \,\mathrm{d}x\right)^{1/n} \leq C(\varepsilon^{\alpha n} + \varepsilon^{\theta(1-\alpha)-\alpha n})^{1/n}.$$

We can choose $\alpha = \theta/(2n+\theta)$ and obtain $v \geq -C\varepsilon^\alpha$. But this implies that $u_* - u_\varepsilon \leq C\varepsilon^\alpha$, which in turn implies that $u_\varepsilon - u \leq C\varepsilon^\alpha$ since $|u_* - u| \leq C\varepsilon^\alpha$.

We finished the proof that $u_\varepsilon - u \leq C\varepsilon^\alpha$. The other inequality follows in the same way. □

We note that even in the case when the solution u to the limiting problem has $C^{2,\delta}$ estimates for some small $\delta > 0$ (as in the convex case) we cannot expect a much better rate of convergence. Indeed, from $u \in C^{2,\delta}(\overline{\Omega})$, we could estimate $u - u_\varepsilon$ at every point $x \in \Omega$. We would have a second order polynomial P_x such that

$$|u(y) - P_x(y)| \leq C|x-y|^{2+\delta}.$$

Therefore $|E_\varepsilon u(x) - F(D^2 u(x))| \leq C\varepsilon^\delta$ for every point x. But from here we would only obtain $|u - u_\varepsilon| \leq \varepsilon^\delta$. On the other hand, in the convex case, if F is smooth ($C^{1,\alpha}$) then u is $C^{3,\alpha}$ (from Schauder estimates on the first derivatives) and we may gain a factor of ε in the rate of convergence in a smooth domain after using this extra regularity.

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