# Regularity estimates for fully non linear elliptic equations which are asymptotically convex

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**Abstract**

In this paper we deliver improved $C^{1,\alpha}$ regularity estimates for solutions to fully nonlinear equations $F(D^2u) = 0$, based on asymptotic properties inherited from its *recession* function $F^\star(M) := \lim_{\mu \to 0} \mu F(\mu^{-1}M)$.

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## 1 Introduction

Regularity estimates for viscosity solution to a given fully nonlinear uniformly elliptic equation

$$F(D^2u) = 0 \text{ in some domain } \Omega \subset \mathbb{R}^n \tag{1.1}$$

has been a primary important line of research since the work of Krylov and Safonov [10, 11] unlocked the theory. By formal linearization, both $u$ and its first derivative, $u_\nu$, satisfy linear elliptic equations in non-divergence form, thus Krylov-Safonov Harnack inequality, implies that solutions are *a priori* $C^{1,\beta}$ for some universal, but unknown $\beta > 0$. The language of viscosity solutions allows the same conclusion without linearizing the equation, see [4]. The question whether a viscosity solution is twice differentiable, i.e. classical, turned out to be truly challenging. The first major result in this direct was obtained independently by Evans [6] and Krylov [8, 9], see also [4, Chapter 6]. This is the contents of the Evans-Krylov $C^{2,\alpha}$ regularity theorem that assures that under concavity or convexity assumption on $F$, viscosity solutions to $F(D^2u) = 0$ are of class $C^{2,\alpha}$ for some $0 < \alpha < 1$. After Evans-Krylov Theorem, many important works attempted to establish a $C^{2,\alpha}$ regularity theory for solutions to special classes of uniform elliptic equations of the form (1.1), see for instance [2] and [19].

Recently Nadirashvili and Vladut, [12, 13] showed that viscosity solutions to fully nonlinear equations may fail to be of class $C^2$. They have also exhibited solutions to uniform elliptic equations whose Hessian blow-up, i.e., that are not $C^{1,1}$. The regularity theory for fully nonlinear equations would turn out to be even more complex: Nadirashvili and Vladut quite recently showed that given any $0 < \tau < 1$ it is possible to build up a uniformly elliptic operator $F$, whose solutions are not $C^{1,\tau}$, see [14, Theorem 1.1]. These examples are made in high dimensions. In [15] and [16], they showed an example of a non $C^2$ solution in five dimensions. This is the lower dimension for which such result is available. In two dimensions, however, it is well known that solutions are always $C^2$. For dimensions $n = 3$ and $n = 4$, the regularity of viscosity solutions to uniformly elliptic equations without further structural assumptions remains an outstanding open problem.

After these stunning examples, it becomes relevant to investigate possible special hidden structures on a given elliptic operator $F$ which might yield further regularity estimates for solutions to (1.1). In this paper we turn attention to an asymptotic property on $F$, called the *recession* function. For any symmetric matrix $M \in \mathbb{R}^{n\times n}$, we define

$$F^\star(M) := \lim_{\mu\to 0} \mu F(\mu^{-1}M). \tag{1.2}$$

The above limit may not exist as $\mu \to 0$. In that case, we say that a recession function $F^\star$ is any one of the subsequential limits.

Heuristically, $F^\star$ accounts the behavior of $F$ at infinity. Recently recession functions appeared in the study of free boundary problems governed by fully nonlinear operators, [17, 1]. The main result we prove in this paper states that the regularity theory for the recession function $F^\star(M)$ grants smoothness of viscosity solutions to the original equation $F(D^2u) = 0$, up to $C^{1,1^-}$.

**Theorem 1.** *Let $F$ be a uniformly elliptic operator. Assume any recession function*

$$F^\star(M) := \lim_{\mu\to 0} \mu F(\mu^{-1}M)$$

*has $C^{1,\alpha_0}$ estimates for solutions to the homogeneous equation $F^\star(D^2v) = 0$. Then, any viscosity solution to*

$$F(D^2u) = 0,$$

*is of class $C^{1,\min\{1,\alpha_0\}^-}_{loc}$. That is, $u \in C^{1,\alpha}_{\text{loc}}$ for any $\alpha < \min\{1, \alpha_0\}$. In addition, there holds*

$$\|u\|_{C^{1,\alpha}(B_{1/2})} \le C\|u\|_{L^\infty(B_1)}, \tag{1.3}$$

*for a constant $C > 0$ that depends only on $n$, $\alpha$ and $F$.*

An immediate Corollary of Theorem 1 is the following:

**Corollary 2.** *Let $F\colon \mathcal{S}(n) \to \mathbb{R}$ be a uniform elliptic operator and $u$ a viscosity solution to $F(D^2u) = 0$ in $B_1$. Assume any recession function $F^\star(M) := \lim_{\mu\to 0} \mu F(\mu^{-1}M)$ is concave. Then $u \in C^{1,\alpha}_{loc}(B_1)$ for every $\alpha < 1$.*

Clearly the corresponding regularity theory for heterogeneous, non-constant coefficient equations $F(X, D^2u) = f(X)$ is, in general, considerably more delicate. Nevertheless, in this setting, L. Caffarelli in [3] established $C^{1,\alpha}$, $C^{2,\alpha}$ and $W^{2,p}$, *a priori* estimates for solutions to

$$F(X, D^2u) = f(X) \in L^p,$$

for $p > n$, under appropriate continuity assumption on the coefficients. Caffarelli's results are, nevertheless, based upon the regularity theory available for the homogeneous, constant-coefficient equation $F(X_0, D^2u) = 0$. Therefore, it is essential to know the best possible regularity estimates available for equations of the form (1.1). Of course, combining Caffarelli's regularity theory and Theorem 1 it is possible to establish the sharp regularity estimates for heterogeneous non-constant coefficient equations.

An application of Corollary 2 concerns local regularity estimates for singular fully nonlinear PDEs:

$$F(D^2u) \sim u^{-\gamma}, \quad 0 < \gamma < 1. \tag{1.4}$$

In [1] it has been proven that nonnegative minimal solutions are locally uniformly continuous and grow precisely as $\text{dist}^{\frac{2}{1+\gamma}}$ away from the free boundary $\partial\{u > 0\}$. Notice that such an estimate implies that $u$ behaves along the free boundary as a $C^{1,\frac{1-\gamma}{1+\gamma}}$ function. In particular, if $\gamma$ is small, such an estimate competes with the (unknown) $C^{1,\alpha_F}$ *a priori* estimate. By knowing the recession function, which governs free boundary condition of the problem, it is possible to show that $u$ is locally of the class $C^{1,\frac{1-\gamma}{1+\gamma}}$ and such an estimate does not deteriorate near the free boundary.

**Corollary 3.** *Let $u$ be a minimal solution to $F(D^2u) \sim u^{-\gamma}$, in $\Omega \subset \mathbb{R}^n$, with $0 < \gamma < 1$. Assume the recession function $F^*$ is unique and has a priori $C^{2,\alpha_*}$ estimates. Then $u$ is locally of class $C^{1,\frac{1-\gamma}{1+\gamma}}$ in $\Omega$.*

The proof of Corollary 3 will be delivered in Section 4. Finally, we would like to point out that Theorem 1 provides eventual gain of smoothness beyond universal estimates only up to $C^{1,1^-}$. Nevertheless, such a constrain does not come from limitations of the methods employed here. In fact, Nadirashvili and Vladut built up an example of a fully nonlinear operator $\mathfrak{F}$ that admits a viscosity solution $\phi \in C^{1,1} \setminus C^2$. Thus, we could deform $\mathfrak{F}$ outside $B_{\|\phi\|_{C^{1,1}}} \subset \mathcal{S}(n)$ as to assure that $\mathfrak{F}^\star$ is linear, say $\mathfrak{F}^\star = \Delta$. Nevertheless, $\phi$ would still be a $C^{1,1} \setminus C^2$ solution to an elliptic equation whose recession function is linear. The final result we prove gives $C^{1,\text{Log-Lip}}$ estimates under the uniform limits and under the assumption that $F^*$ has *a priori* $C^{2,\alpha_\star}$ interior estimates. More precisely we have

**Theorem 4.** *Let $F\colon \mathcal{S}(n) \to \mathbb{R}$ be a uniform elliptic operator and $u$ a viscosity solution to $F(D^2u) = f \in \text{BMO}$ in $B_1$. Assume the recession function $F^\star(M) := \lim\limits_{\mu \to 0} \mu F(\mu^{-1}M)$ exists and has a priori $C^{2,\alpha_\star}$ interior estimates. Assume further that the limit $\lim\limits_{\mu \to 0} \mu F(\mu^{-1}M)$ is uniform in $M$. Then $u \in C^{1,Log\text{-}Lip}_{loc}(B_1)$, i.e.,*

$$|u(X) - [u(X_0) + \nabla u(X_0) \cdot (X - X_0)]| \leq -C|X - X_0|^2 \log |X - X_0|.$$

## 2 Preliminaries

In this section make few comments about the notion of *recession* function. Throughout this paper, $B_r$ denotes the ball of radius $r > 0$ in the Euclidean space $\mathbb{R}^n$ and $\mathcal{S}(n)$ denotes the space of all real, $n \times n$ symmetric matrices. A function $F\colon \mathcal{S}(n) \to \mathbb{R}$ will always be a uniformly elliptic operator, as in [4]. That is, we assume that there exist two positive constants $0 < \lambda \leq \Lambda$ such that, for any $M \in \mathcal{S}(n)$ there holds

$$\lambda \|P\| \leq F(M + P) - F(M) \leq \Lambda \|P\|, \quad \forall P \geq 0. \tag{2.1}$$

We will further assume, with no loss of generality, that $F(0) = 0$.

A key information we shall use in the proof of Theorem 1 is the fact that solutions to (1.1) are locally $C^{1,\epsilon}$ for some universal $\epsilon > 0$. Furthermore

$$\|u\|_{C^{1,\epsilon}(B_{1/2})} \leq C\|u\|_{L^\infty(B_1)},$$

for a universal constant $C > 0$. As mentioned in the introduction, Nadirashvili and Vladut have proven that $C^{1,\epsilon}$ is the best regularity theory available for general fully nonlinear elliptic equations. The objective of this paper is to show that further smoothness could be assured if we have information on the recession function of $F$, defined in (1.2)

Let us discuss a bit about recession functions for fully nonlinear elliptic equations. Initially, it is straightforward to verify that for each $\mu$, the elliptic operator

$$F_\mu(M) := \mu F(\mu^{-1}M)$$

is uniformly elliptic, with the same ellipticity constants as $F$. Thus, up to a subsequence, $F_\mu$ does converge to a limiting elliptic operator $F^\star$ as $\mu \to 0$. Any limiting point $F^\star$ will be called a *recession* function of $F$. This terminology comes from the theory of Haminton-Jacobi equations, see for instance, [7].

Initially, let us point out that recession functions may not be unique, as simple 1-d examples show. Nevertheless, if the recession function is unique, it is clearly homogeneous of degree one, that is, for any scalar $t$, we have

$$F^\star(tM) = tF^\star(M).$$

Also, if $F$ is homogeneous of degree one, then $F = F^\star$. In some applications, it is possible to verify that

$$\lim_{\|M\|\to\infty} D_{i,j}F(M) =: F_{ij}. \tag{2.2}$$

That is, $F$ has a linear behavior at the ends. Under such condition, it is simple to check that $F^\star$ is a linear elliptic operator, and, in fact,

$$F^\star(M) = \mathrm{tr}\,(F_{ij}M_{ij})\,.$$

A particularly interesting example is the class of Hessian operators of the form

$$F_\iota(M) = f_\iota(\lambda_1, \lambda_2, \cdots \lambda_n) := \sum_{j=1}^{n}(1+\lambda_j^\iota)^{1/\iota},$$

where $\iota$ is an odd natural number. For this family of operators, we have

$$F_\iota^\star = \Delta.$$

A priori $F_\mu$ converges pointwisely to $F^\star$. However, the following is a more precise description of how the limit takes place.

**Lemma 5.** *If $F$ is any uniformly elliptic operator and $F^\star(M) = \lim_{\mu\to 0}\mu F(\mu^{-1}M)$ for every symmetric matrix $M$, then for every $\varepsilon >$, there exists a $\delta > 0$ so that*

$$||\mu F(\mu^{-1}M) - F^*(M)|| \le \varepsilon(1 + ||M||), \tag{2.3}$$

*for all $\mu < \delta$.*

*Proof.* Since the function $F$ is uniformly elliptic, we have that $F(X+Y) - F(X) \le \Lambda||Y||$ for some constant $\Lambda$ and $F$ is Lipschitz. This Lipschitz norm is conserved by the scaling $\mu F(\mu^{-1}M)$. By the Arzela-Ascoli theorem we have that up to a subsequence $\mu F(\mu^{-1}M)$ converges uniformly in every compact set. Since $\mu F(\mu^{-1}M)$ converges pointwise to $F^*$, then all its subsequential limits must coincide with $F^\star$ and therefore it converges to $F^\star$ uniformly over every compact set.

That means that for every $\varepsilon > 0$ there exists a $\delta > 0$ so that

$$||\mu F(\mu^{-1}M) - F^\star(M)|| \le \varepsilon,$$

for all matrices $M$ such that $||M|| \le 1$ and all $\mu < \delta$. This already shows that (2.3) holds if $||M|| \le 1$.

Now let $M$ be a matrix with $||M|| > 1$. For any $\mu < \delta$, we can consider also $\mu_1 = ||M||^{-1}\mu < \mu < \delta$. Therefore

$$\left\|\mu_1 F\left(\mu_1^{-1}\frac{M}{||M||}\right) - F^\star\left(\frac{M}{||M||}\right)\right\| \le \varepsilon,$$

Observing that $\mu_1^{-1}\frac{M}{||M||} = \mu^{-1}M$, and using that $F^*$ is homogeneous of degree one, we obtain

$$\left\|\mu F\left(\mu^{-1}M\right) - F^\star\left(M\right)\right\| \le \varepsilon ||M||.$$

This proves (2.3) for $||M|| > 1$. □

A model case though is when $F$ equals $F^\star$ outside a ball $B_R \subset \mathrm{Sym}(n)$, for some $R \gg 1$. In this case, the convergence $F_\mu \to F^\star$ is uniform with respect to $M$ – compare with the hypothesis of Theorem 4.

## 3 $C^{1,\alpha_0^-}$ estimates

In this section we prove Theorem 1. We start off the proof by fixing an aimed Hölder continuity exponent for gradient of $u$ between 0 and $\alpha_0$, more precisely, we fix

$$0 < \alpha < \min\{1, \alpha_0\}. \tag{3.1}$$

We will show that $u \in C^{1,\alpha}$ at the origin. It is standard to pass from pointwise estimate to interior regularity. Initially, as mentioned in the introduction, it follows from Krylov-Safonov Harnack inequality that $u \in C^{1,\epsilon}$ for some universal $\epsilon > 0$. We may assume, therefore, by normalization and translation, that

$$|u| \le 1 \text{ in } B_{9/11} \tag{3.2}$$

$$u(0) = |\nabla u(0)| = 0. \tag{3.3}$$

Our strategy is based on the following reasoning: proving that $u \in C^{1,\alpha}$ at the origin is equivalent to verifying that either there exists a constant $C > 0$ such that

$$\sup_{B_r} |u(X)| \le C r^{1+\alpha}, \quad \forall r < 1/5,$$

or else, by iteration, that for some $\ell > 0$ and some $r > 0$, there holds

$$\sup_{B_r} |u(X)| \le 2^{-(1+\alpha)\ell} \sup_{B_{2^\ell \cdot r}} |u(X)|,$$

see [5], Lemma 3.3 for similar inference. Therefore, if we suppose, for the purpose of contradiction, that the thesis of the Theorem fails, there would exist a sequence of viscosity solutions $F(D^2 u_k) = 0$, satisfying (3.2) and (3.3), and a sequence of radii $r_k \to 0$ such that

$$(\sup_{B_{r_k}} |u_k|)^{-1} \cdot r_k^{(1+\alpha)} \longrightarrow 0 \tag{3.4}$$

$$\sup_{B_{r_k}} |u_k| \geq 2^{-(1+\alpha)\ell} \sup_{2^\ell \cdot r_k} |u_k|. \tag{3.5}$$

For notation convenience, let us label

$$s_k := \sup_{B_{r_k}} |u_k|.$$

In the sequel, we define the normalized function

$$v_k(X) := \frac{1}{s_k} u_k(r_k X).$$

Immediately, from definition of $v_k$, we have

$$\sup_{B_1} |v_k| = 1. \tag{3.6}$$

Also, it follows from (3.5) that $v_k$ growths at most as $|X|^{1+\alpha}$, i.e.,

$$\sup_{B_{2^\ell}} v_k \leq 2^{(1+\alpha)\ell}. \tag{3.7}$$

In addition, if we define the uniform elliptic operator

$$F_k(M) := (s_k^{-1} \cdot r_k^2) F\left((s_k \cdot r_k^{-2}) M\right),$$

we find out that $v_k$ solves

$$F_k\left(D^2 v_k\right) = 0, \tag{3.8}$$

in the viscosity sense. By uniform ellipticity and (3.4), up to a subsequence, $F_k$ converges locally uniformly to a recession function $F^\star$. Thus, letting $k \to \infty$, by $C^{1,\epsilon}$ universal estimates, $v_k \to v_\infty$ locally in the $C^{1,\epsilon/2}(\mathbb{R}^n)$ topology. Clearly $v_\infty$ is a viscosity solution to

$$F^\star(D^2 v_\infty) = 0 \text{ in } \mathbb{R}^n.$$

Taking into account (3.3), (3.6), (3.7), we further conclude that $v_\infty$ satisfies

$$\begin{aligned} v_\infty(0) = |\nabla v_\infty(0)| &= 0, && (3.9) \\ \sup_{B_1} |v_\infty| &= 1, && (3.10) \\ |v_\infty(Y)| &\leq |Y|^{1+\alpha}. && (3.11) \end{aligned}$$

Hereafter let us label

$$\kappa := \min\{1, \alpha_0\} - \alpha > 0.$$

Recall any recession function $F^\star$ is homogeneous of degree one for positive multipliers. Therefore, fixed a large positive number $\ell \gg 1$, the auxiliary function

$$\mathcal{V}_\infty(Z) := \frac{v_\infty(\ell Z)}{\ell^{1+\alpha}},$$

too satisfies

$$F^\star(D^2\mathcal{V}_\infty) = 0.$$

From (3.11) we verify that $\mathcal{V}_\infty$ is bounded in $B_1$ and, hence, from the regularity theory for the recession function, $F^\star$, there exists a constant $C^\star$, depending on dimension and $F^\star$, such that

$$|\nabla \mathcal{V}_\infty(Z)| \le C^\star |Z|^{\alpha+\kappa}, \quad \forall Z \in B_{1/5}. \tag{3.12}$$

Finally, estimate (3.12) gives, after scaling,

$$\begin{aligned} \sup_{B_{\frac{\ell}{5}}} \frac{|\nabla v_\infty(Y)|}{|Y|^{\alpha+\kappa}} &= \ell^{-\kappa} \sup_{B_{\frac{1}{5}}} \frac{|\nabla \mathcal{V}_\infty(Z)|}{|Z|^{\alpha+\kappa}} \\ &= \mathrm{o}(1), \end{aligned} \tag{3.13}$$

as $\ell \to \infty$. Clearly (3.13) implies that $v_\infty$ is constant in the whole $\mathbb{R}^n$. However, such a conclusion drives us into a contradiction, since, from (3.9), $v_\infty \equiv 0$ which is incompatible with (3.10). The proof of Theorem 1 is concluded.

# 4 Proof of Corollary 3

In this Section we comment on the proof of Corollary 3. Given a point $X \in \{u > 0\}$, with

$$d := \mathrm{dist}(X, \partial\{u > 0\}) < \frac{1}{2}\mathrm{dist}(X, \partial\Omega),$$

we consider $Y \in \partial\{u > 0\}$, such that $d = |X - Y|$. Applying Corollary 2 we can estimate

$$[u]_{C^{1,\frac{1-\gamma}{1+\gamma}}(B_{d/4}(X))} \lesssim \frac{1}{d^{\frac{2}{1+\gamma}}} \left( \|u\|_{L^\infty(B_{d/2}(Z))} + d^2 \cdot \|u^{-\gamma}\|_{L^\infty(B_{d/2}(Z))} \right). \tag{4.1}$$

It then follows by the optimal control

$$u(\xi) \sim \mathrm{dist}(\xi, \partial\{u > 0\})^{\frac{2}{1+\gamma}},$$

see [1, Theorem 9], that we can estimate, in $B_{d/2}(Z)$,

$$\|u\|_{L^\infty(B_{d/2}(Z))} \lesssim d^{\frac{2}{1+\gamma}}, \tag{4.2}$$

$$\|u^{-\gamma}\|_{L^\infty(B_{d/2}(Z))} \lesssim d^{\frac{-2\gamma}{1+\gamma}}. \tag{4.3}$$

Plugging (4.2) and (4.3) into (4.1) gives

$$[u]_{C^{1,\frac{1-\gamma}{1+\gamma}}(B_{d/4}(X))} \lesssim 1,$$

and therefore $u$ is locally of class $C^{1,\frac{1-\gamma}{1+\gamma}}$, up to the free boundary. The proof is complete. □

## 5 Proof of Theorem 4

For this section we assume that $\lim_{\mu\to 0}\mu F(\mu^{-1}M)$ exists and equals $F^*(M)$ for every matrix $M$. It is also part of our assumption that the limit is uniform in $M$. In particular, given $\varepsilon > 0$, we can find $\delta > 0$, such that

$$|F_\mu(M) - F(M)| \le \varepsilon, \quad \forall M$$

provided $0 < \mu \le \delta$.

**Lemma 6.** *Assume $F$ and $F^*$ are two fully nonlinear uniformly elliptic operators such that*

$$|F(M) - F^*(M)| \le \varepsilon, \qquad \text{for any symmetric matrix } M. \tag{5.1}$$

*Assume moreover that $F^*(0) = 0$ and $F^*$ has $C^{2,\alpha_*}$ estimates in the form that any solution $u^*$ of $F^*(D^2u^*) = 0$ in $B_1$ satisfies*

$$||u^*||_{C^{2,\alpha_*}(B_{1/2})} \le C_*||u||_{L^\infty(B_1)}. \tag{5.2}$$

*Then there exist two constants $\varepsilon$ and $r$ (depending only on the ellipticity constants, dimension, $C_*$ and $\alpha_*$) so that for any solution $u$ of $F(D^2u) = f$ in $B_1$ with $||f||_{L^\infty} \le \varepsilon$ and $||u||_{L^\infty} \le 1$, there exists a second order polynomial $P$, such that $||P|| \le C$ and $||u - P||_{L^\infty(B_r)} \le r^2$.*

*Proof.* The value of $r$ will be specified below in terms of the $C^{2,\alpha_*}$ estimate (5.2) only. For that value of $r$, we prove the lemma by contradiction. If the result was not true, there would exist a sequence $F_n$, $F_n^*$ $f_n$, $u_n$ so that

$$\begin{aligned}
|F_n(M) - F_n^*(M)| &\le \frac{1}{n} \qquad \text{for any symmetric matrix } M,\\
||f_n||_{L^\infty(B_1)} &\le \frac{1}{n},\\
||u_n||_{L^\infty(B_1)} &\le 1,\\
F_n(D^2u_n) &= f_n \qquad \text{in } B_1,\\
F_n \text{ and } F_n^* &\text{ have uniform ellipticity constants } \lambda, \Lambda,
\end{aligned}$$

where $F_n^*$ has $C^{2,\alpha_*}$ estimates as in (5.2) but such polynomial $P$ cannot be found for any $u_n$.

Since the $F_n^*$ are uniformly elliptic, in particular they are uniformly Lipschitz. Up to extracting a subsequence, they will converge to some uniformly elliptic function $F^*$ which will also have $C^{2,\alpha_*}$ estimates (5.2). Thus, we can assume that all $F_n^*$ are the same by replacing them by $F^*$ (and taking a subsequence if necessary).

Since the $F_n$ are uniformly elliptic, the functions $u_n$ are uniformly $C^\alpha$ in the interior of $B_1$ and there must be a subsequence that converges locally uniformly to some continuous function $u_*$. We extract this subsequence, and by abuse of notation we still call it $u_n$. Since $u_n \to u_*$ locally uniformly, $F_n \to F^*$ locally uniformly, and $f_n \to 0$ uniformly, we have that $F^*(D^2u^*) = 0$ holds in the viscosity sense. From the $C^{2,\alpha_*}$ estimates (5.2), if we choose $P$ to be the second order Taylor's expansion of $u^*$ at the origin we will have

$$||u^* - P||_{L^\infty(B_r)} \leq C_* r^{2+\alpha_*}.$$

We choose $r$ small enough so that $C_* r^{\alpha_*} < 1/2$. Note that this choice depends on $C_*$ and $\alpha_*$ only. We thus have

$$||u^* - P||_{L^\infty(B_r)} \leq \frac{r^2}{2}.$$

However, since $u_n \to u_*$ uniformly in $B_r$, then for $n$ large enough we also have.

$$||u_n - u^*||_{L^\infty(B_r)} \leq \frac{r^2}{2}.$$

Combining the last two previous inequalities we obtain that

$$||u_n - P||_{L^\infty(B_r)} \leq r^2,$$

and so we arrive to a contradiction since we were assuming that such polynomial $P$ did not exist for any $n$. □

*Proof of Theorem 4.* We prove the result for $x_0 = 0$ and assuming $f \in L^\infty$ – see [18] for the adjustments requested when $f \in \mathrm{BMO}$. From uniform convergence hypothesis, we can find $\delta > 0$ so that for all $\mu < \delta$ the inequality

$$||\mu F(\mu^{-1}M) - F^\star(M)|| \leq \varepsilon,$$

holds, where $\varepsilon > 0$ is the number from Lemma 6. We start off now with a convenient rescaling of the problem. We find an $r_0$, depending only on $||u||_{L^\infty}$ and $\delta$, and consider the scaling

$$u_0(x) = \varepsilon \max\{1, \|u\|_{L^\infty}, \|f\|_\infty\}^{-1} u(r_0 x).$$

We choose $r_0 \sim \sqrt{\delta}$, where $\delta$ is the number above. For this choice we have

$$\|u_0\|_{L^\infty(B_1)} \leq 1;$$
$$\mu F(\mu^{-1} D^2 u_0) = \tilde{f}(x),$$

for a $\mu < \delta$ and $\|\tilde{f}\|_\infty \leq \varepsilon$. Now we proceed to show that $u_0$ is $C^{1,\text{Log-Lip}}$ at the origin. The strategy is to show the existence of a sequence of quadratic polynomials

$$P_k(X) := a_k + \mathbf{b}_k \cdot X + \frac{1}{2} X^t M_k X,$$

where, $P_0 = P_{-1} = 0$, and for all $k \geq 0$,

$$F^\star(M_k) = 0, \qquad (5.3)$$

$$\sup_{Q_{r^k}} |u_0 - P_k| \leq r^{2k}, \qquad (5.4)$$

$$|a_k - a_{k-1}| + r^{k-1}|\mathbf{b}_k - \mathbf{b}_{k-1}| + r^{2(k-1)}|M_k - M_{k-1}| \leq Cr^{2(k-1)}. \qquad (5.5)$$

The radius $r$ in (5.4) and (5.5) is the one from Lemma 6. We shall verify (5.3)–(5.5) by induction. The first step $k = 0$ is immediately satisfied. Suppose we have verified the thesis of induction for $k = 0, 1, \cdots, i$. Define the re-scaled function $v\colon B_1 \to \mathbb{R}$ by

$$v(X) := \frac{(u_0 - P_i)(r^i X)}{r^{2i}},$$

It follows by direct computation that $v$ satisfies $|v| \leq 1$ and it solves

$$\mu F\left(\mu^{-1}\left(D^2 v + M_i\right)\right) = \tilde{f}(r^i x).$$

If we define

$$F_i(M) := F(M + M_i) \quad \text{and} \quad F_i^\star(M) := F^\star(M + M_i),$$

it follows from uniform convergence that

$$F_i \text{ is close to } F_i^\star.$$

Furthermore, since $F^\star(M_i) = 0$, the homogeneous equation

$$F_i^\star(D^2\xi) = 0$$

satisfies the same conditions as the original $F^\star$. We now apply Lemma 6 to $v$ and find a quadratic polynomial $\tilde{P}$ such that

$$\|v - \tilde{P}\|_{L^\infty(B_r)} \leq r^2. \qquad (5.6)$$

If we define

$$P_{i+1}(X) := P_i(X) + r^{2i}\tilde{P}(r^{-i}X)$$

and rescale (5.6) back, we conclude the induction thesis. In the sequel, we argue as in [18]. From (5.5) we conclude that $a_k \to u_0(0)$ and $\mathbf{b}_k \to \nabla u_0(0)$, in addition

$$|u_0(0) - a_k| \leq C\rho^{2k} \qquad (5.7)$$

$$|\nabla u_0(0) - \mathbf{b}_k| \leq C\rho^k. \qquad (5.8)$$

From (5.5) it is not possible to assure convergence of the sequence of matrices $(M_k)_{k\geq 1}$; nevertheless, we estimate

$$|M_k| \leq Ck. \tag{5.9}$$

Finally, given any $0 < r < 1/2$, let $k$ be the integer such that

$$\rho^{k+1} < r \leq \rho^k.$$

We estimate, from (5.7), (5.8) and (5.9),

$$\begin{aligned} \sup_{Q_r} |u(X) - [u(0) + \nabla u(0) \cdot X]| &\leq \rho^{2k} + |u(0) - a_k| + \rho|\nabla u(0) - \mathbf{b}_k| \\ &+ \rho^{2k}|M_k| \\ &\leq -Cr^2 \log r, \end{aligned}$$

and the Theorem is proven.

□

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