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Problems in infinite loop space theory

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This will be a rambling discussion of the status of infinite loop space theory, with the emphasis on unsolved problems in geometric topology, algebraic K-theory, cobordism, homology operations, and the general abstract theory. Many of the problems will involve E_∞ ring spaces and spectra. This is not solely a reflection of my personal interests. Implicitly or explicitly, these structures are central to most of the applications. The basic reference is [9], which will contain the material of the preprints "Coordinate-free spectra", "$\mathcal{J}$-functors and orientation theory", "E_∞ ring spectra" (with Frank Quinn and Nigel Ray), and "On kO-oriented bundle theories", as well as the material on E_∞ ring spaces about which I talked at the conference and other material developed since (some of which is sketched below).

We now have a coherent theory of infinite loop spaces and spectra, and it has gradually become apparent what the theory can and cannot do. I know of very few significant examples of H-spaces which are suspected but not known to be infinite loop spaces and there are no outstanding theoretical problems concerning the recognition of infinite loop spaces. The only possible exception might be Quillen's old conjecture that the representing space of a functor with an appropriate transfer is an infinite loop space. It is by now well understood that this conjecture is wholly implausible, but that any counter-example would be sufficiently ugly as to be wholly uninteresting (see Lada [4]).

The situation as regards the recognition of E_∞ ring spectra is not quite as satisfactory. There are significant examples of ring spectra which are suspected but not known to be E_∞ ring spectra. The only Thom spectrum not yet accounted for is MPL (and Quinn is working on this). A very provocative example is BP.

Problem 1. Does the Brown-Peterson spectrum admit a model as an E_∞ ring spectrum ?

The point here is that the notion of an E_∞ ring spectrum seems not to be a purely homotopical one; good concrete geometric models are required, and no such model is known for BP. For much the same reason, we have the awkward state of affairs revealed by the following question.

Problem 2. Are localizations and completions of E_∞ ring spectra again E_∞ ring spectra?

Of course, we can use global E_∞ ring spectra to obtain infinite loop space information and can then localize or complete (as must be done even to make sense of some of the geometric problems below).

One way to handle the local version of Problem 2 would be to carry out the following program, which is certainly well within reach and should have other useful applications.

Problem 3. Develop recognition principles for passage from "p local" E_∞ spaces and E_∞ ring spaces to p local spectra and E_∞ ring spectra.

The idea here is to reconstruct the entire machine of [7, 8, 9] with the full symmetric groups replaced by suitably compatible p-Sylow subgroups.

The situation as regards the recognition of infinite loop maps is still less satisfactory. The machine does produce lots of infinite loop maps and does show that once certain key maps are proven to be infinite loop maps then it will follow that various other maps are so as well. Unfortunately, many of the key maps have yet to be proven to be infinite loop maps. I suspect that this is in the general nature of things and that improvements in the abstract theory will be of little help with the remaining problems. However, a solution to Problem 3 would be of some use since the machine has a marked aversion to virtual representations and the localization at p of a global map defined in terms of virtual representations can sometimes be defined in terms of honest representations.

Of course, we need not rely only on naturality properties of the machine to construct infinite loop maps, since all of the familiar techniques of stable homotopy theory are also available. There is as yet no analog of this statement for maps of E_∞ ring spectra: the only technique currently available is the construction of models acceptable to the machine.

Given this picture of the abstract situation, let's turn to concrete applications. We would like to have a complete understanding of Adams' analysis of the groups JO(X) and of Sullivan's analysis of KTop(X) away from 2 on the level of cohomology theories or, more or less equivalently, on the level of infinite loop space structures on the classifying spaces of various fibration and bundle theoretic functors. Explicitly, we would like to decompose the localizations or completions of all relevant infinite loop spaces into products (or, at 2, fibrations) of specific atomic pieces, namely BCoker J and pieces obtained solely from BO (such as B Im J, the various factors of BO at odd primes, BSpin, etc.). The following three conjectures would supply the key infinite loop maps necessary for this purpose.

<u>Conjecture 1.</u> The complex Adams conjecture holds on the infinite loop space level.

This means that, for each r, the composite

$$BU \xrightarrow{\psi^r - 1} BU \xrightarrow{Bj} BSF$$

is trivial as an infinite loop map when localized away from r. The real analog is wildly false. With r = 3 and U replaced by O, Madsen [5] showed that not even the first delooping is null homotopic. The splitting of BSF as an infinite loop space at odd primes (Tornehave [16] or [9]) shows that the conjecture is at least plausible, and several people seem to be working on it.

To state the next two conjectures, we require some preliminaries from [9]. Let G be a bundle theory (O, U, Spin, Top, F, etc.). Let B(G; E) be the classifying space for E-oriented stable G-bundles and let $q: B(G;E) \to BG$ correspond to neglect of orientation. An E-orientation of G is an H-map $g: BG \to B(G;E)$ such that $q \circ g \simeq 1$. Thus g specifies natural E-orientations, with product formula, of stable G-bundles. When E is an E_∞ ring spectrum, B(G; E) is an infinite loop space. Say that g is perfect if it is an infinite loop map and if $q \circ g \simeq 1$ as infinite loop maps.

<u>Conjecture 2.</u> The Atiyah-Bott-Shapiro orientation $g: BSpin \to B(Spin; kO)$ is perfect.

<u>Conjecture 3.</u> The Sullivan orientation $\bar{g}: BSTop \to B(STop; kO[1/2])$ is perfect.

Quinn is working on the second of these.

To relate these conjectures to BCoker J, we compose g and $\bar{g}$ with the natural infinite loop maps from their ranges to $B(SF; kO)$ (suitably localized) and then apply the universal cannibalistic class $c(\psi^r): B(SF; kO) \to BSpin_{\otimes}$, where $BSpin_{\otimes}$ is the 2-connective cover of the special unit infinite loop space $BO_{\otimes} = SF\, kO$ (see [9]). At p, the fibre of $c(\psi^{r(p)})$ is B Coker J, where $r(2) = 3$ and, for $p > 2$, $r(p)$ is a power of a prime $q \neq p$ such that $r(p)$ reduces mod p^2 to a generator of the group of units of Z_{p^2}. An affirmative answer to the following question would imply that $c(\psi^r)$ is an infinite loop map.

<u>Problem 4.</u> Is $\psi^r: kO[1/r] \to kO[1/r]$ a map of E_∞ ring spectra?

As our abstract discussion makes clear, we are not yet close to an answer. To avoid this question, we turn to discrete models and algebraic K-theory. Recall from [9] that "bipermutative categories" naturally give rise to E_∞ ring spaces and thus to E_∞ ring spectra. For a commutative topological ring A, we have bipermutative categories $\mathcal{O}A$ and $\mathcal{GL}A$ of orthogonal and general linear groups. The E_∞ ring spectrum kO used above is obtained from $\mathcal{O}\mathbb{R}$. Let kO^δ denote the completion away from q (as above, when thinking of p) of the E_∞ ring spectrum obtained from $\mathcal{O}\bar{k}_q$, where $\bar{k}_q$ is an algebraic closure of the field of q elements. Brauer lifting yields an equivalence $\hat{\lambda}: kO^\delta \to \hat{kO}[1/q]$ of ring spectra. A local version of this result (in the complex case) was proven by Tornehave [15]; on the completed level (which is the one of topological interest), the result is in fact extremely simple [9].

Problem 5. Is $\hat{\lambda}: kO^\delta \to \hat{kO}[1/q]$ a map of E_∞ ring spectra?

Again, we are not yet close to an answer. The Frobenius automorphism $\phi^r: \mathcal{O}\bar{k}_q \to \mathcal{O}\bar{k}_q$ $(r = q^a)$ is a morphism of bipermutative categories, and the induced map (again denoted ϕ^r) on kO^δ is transported to ψ^r via $\hat{\lambda}$. $c(\phi^r): kO^\delta \to BSpin^\delta_\otimes$ (the 2-connective cover of $BO^\delta_\otimes = SF\, kO^\delta$) is an infinite loop map. Its fibre at p, with $r = r(p)$, is BCoker J endowed with an infinite loop space structure. This fibre admits a more conceptual description as $B(SF; jO^\delta)$, where jO^δ is a certain E_∞ ring spectrum which is a discrete model for the fibre jO of $\psi^{r(p)} - 1: kO \to kSpin$ at p (see [9]). In view of Problems 4 and 5, it should be apparent that, at present, Conjectures 2 and 3 would be much more useful if proven with kO replaced by kO^δ. So reformulated, they and Conjecture 1 would complete the desired analysis of Adams' work and of Sullivan's work away from 2. The sketch above is philosophically sound because kO^δ and $\hat{kO}[1/q]$ are indistinguishable on the motivating level of multiplicative cohomology theories and because $BO^\delta_\otimes$ and $\hat{BO}_\otimes[1/q]$ are equivalent as infinite loop spaces at $p \neq q$.

Here the last clause follows from a recent result of Adams and Priddy which shows that, up to equivalence, there is only one p-local or p-complete connective spectrum with zeroth space equivalent to BSO localized or completed at p. Again, maps are more difficult.

Problem 6. When is an H-map between two infinite loop spaces, both equivalent to BSO localized or completed at p, an infinite loop map ?

Adams's map $\rho^3: BSO \to BSO_{\otimes}$ is a good test case. The cannibalistic classes $\rho^r: BSpin \to BSpin_{\otimes}$ will be infinite loop maps when completed away from r if Conjecture 2 holds (in its original form) and (a weakened form of) either Problem 4 or Problem 5 has an affirmative answer. The Adams-Priddy result shows that F/Top and $BO_{\otimes}$ are equivalent as infinite loop spaces at each $p > 2$. If Conjecture 3 holds (in its original form), then the Sullivan L-genus equivalence $F/Top \to BO_{\otimes}$ away from 2 will be an infinite loop map.

We are still very far from understanding F/Top at $p = 2$, and the following problem is probably beyond reach at present.

Problem 7. Describe F/Top and F/PL as infinite loop spaces at $p = 2$; then describe BTop and BPL.

Madsen [5] showed that $B^3(F/Top)$ does not split at 2 as a product of Eilenberg-MacLane spaces; Madsen and Milgram [6] have shown that $B^2(F/Top)$ does so split.

We return for a moment to the categories $\mathcal{GL}A$ and $\mathcal{O}A$ for a discrete commutative ring A. Let kA and kOA denote the resulting E_∞ ring spectra. Quillen's algebraic K-groups are $K_iA = \pi_i kA$ for $i \geq 1$ (and the E_∞ ring structure gives K_*A a ring structure). Quillen's work suggests that it may be reasonable to define $KO_iA = \pi_i kOA$ for $i \geq 1$ and to regard the natural map $KO_*A \rightarrow K_*A$ as analogous to complexification.

Problem 8. What are the images of the stable stems in KO_*Z and K_*Z under the induced maps on homotopy groups of the units e of the spectra kOZ and kZ?

Quillen showed that, in degrees 4s-1, the image of J maps monomorphically to K_*Z (onto a direct summand except possibly for 2-torsion when s is odd), and the image of J maps monomorphically onto a direct summand of KO_*Z in all degrees [9]. Beyond the obvious stable families of order 2, which map monomorphically onto direct summands of both KO_*Z and K_*Z, nothing is known about the behavior of the cokernel of J. While one really wants to know all of K_*Z, such a calculation seems unlikely to come out of infinite loop space techniques.

Problem 9. What is the precise relationship between the machine-built spectrum kA and the Gersten-Wagoner spectrum KA [2, 18]?

One would hope that, modulo adjustment necessitated by K_0A, kA is the connective spectrum associated to KA. This problem, and the category $\mathcal{O}A$, are closely related to Hermitian K-theory and algebraic L-theory, an area which abounds in bipermutative categories whose associated E_∞ ring spectra have yet to be studied. A good solution of Problem 9 should also yield the conjecture of Karoubi [3, p. 397 and 392] in the topological form stated by Wall [19, p. 292]. Incidentally, the theory of E_∞ ring spectra should answer the question about products raised in [19, p. 292] although here, and in various other applications, a solution of the following problem may eventually be required in order to obtain a really complete picture.

<u>Problem 10.</u> Develop a theory of E_∞ pairings of E_∞ module spectra over an E_∞ ring spectrum.

We turn next to homological calculations in geometric topology. On the infinite classifying space level, the calculations are quite complete (and will be summarized in [11]), although we still know relatively little about how to interpret them.

<u>Problem 11.</u> Find fibration-theoretic interpretations of characteristic classes for spherical fibrations.

Ravenel [13] and others have given such interpretations of certain classes, but not enough to generate $H^*(BSF)$ (under all structure in sight, including the duals of the homology operations).

We can read off $H^*(BSF(n); Z_p)$ from $H^*(BSF; Z_p)$ when $p = 2$ or n is odd, and similarly for $G(n+1)$. The following problem should not be too difficult.

Problem 12. Compute $H_*(BSF(n); Z_p)$ and $H_*(BSG(n+1); Z_p)$ for p odd and n even.

Provided that $PL(n)$ and $Top(n)$ are interpreted in the (comparatively uninteresting) block bundle sense, $G(n)/PL(n) \simeq G/PL$ for $n \geq 3$ and $Top(n)/PL(n) \simeq Top/PL$ for $n \geq 5$. Thus the following problem should be solvable by comparison with the infinite case.

Problem 13. Compute $H^*(BPL(n); Z_p)$ and $H^*(BTop(n); Z_p)$ for all primes p.

The analog for the usual $PL(n)$ and $Top(n)$ is still beyond reach (and iterated loop space techniques probably have little relevance).

On the Thom spectrum level, there is still much to be done. At the prime 2, Brumfiel, Madsen, and Milgram [1] in the unoriented case and Madsen and Milgram [unpublished] in the oriented case have obtained essentially complete information about Top and PL cobordism. At odd primes, Tsuchiya [17] showed that the kernel of the natural map $A \to H^*(MSTop)$ is the left ideal generated by the Milnor elements Q_0 and Q_1. Unfortunately, that now seems to be the easiest step in the following program.

Problem 14. Compute $H^*(MSTop; Z_p)$ as a module over the Steenrod algebra A; then compute $\pi_* MSTop$ by use of the Adams spectral sequence; find representative manifolds for the resulting cobordism classes.

See Peterson [12] for a possible approach to the first step. The following easier problem is also still open.

Problem 15. Compute $\pi_* MSF$ explicitly; use the Levitt exact sequence to read off the oriented Poincaré duality cobordism groups and find representative Poincaré complexes for the resulting cobordism classes.

Since MSF splits as a product of Eilenberg-MacLane spectra (see Peterson [12]), the first step (on the additive level) requires only a counting argument from the known structure of $H^*(BSF)$.

The previous problems deal primarily with the cohomology of spectra MG for stable bundle theories G. Since the MG are E_∞ ring spectra, their zeroth spaces M_0G are E_∞ ring spaces and their unit spaces $FMG \subset M_0G$ are infinite loop spaces. The infinite loop map $Be: BF \rightarrow BFMG$ is the universal obstruction to the MG-orientability of stable spherical fibrations [9]. Even when MG is just a product of Eilenberg-MacLane spectra, the spectrum determined by FMG may well be complicated. The following would be a first step towards understanding these spectra. Define an AR-Hopf bialgebra (with χ) to be an A-coalgebra together with two structures of R-algebra (and a conjugation χ for the additive structure) subject to all requisite commutation formulas between the various pieces of structure [10, 11]. (The less appropriate term "Hopf ring" has been used by other authors.)

<u>Problem 16.</u> Compute $H_*(M_0G; Z_p)$ as an AR-Hopf bialgebra; then compute $H_*(BFMG; Z_p)$.

When $G = \{e\}$, MG is the sphere spectrum and $H_*(M_0G; Z_p)$ is the free AR-Hopf bialgebra with χ generated by H_*S^0 because the "Mixed Cartan formula" and "mixed Adem relations" completely determine the multiplicative homology operations in terms of the additive homology operations [11]. Similarly, the free AR-Hopf bialgebra (without χ) is realized by $H_*(\coprod B\Sigma_n; Z_p)$.

Ravenel and Wilson [14] have computed $H_*(M_0U; Z_p)$ as a Hopf bialgebra (without homology operations).

While the theory of homology operations on E_∞ ring spaces is well understood, there should also be a related theory of homotopy operations.

<u>Problem 17.</u> Analyze the homotopy operations implicit in the definition of E_∞ ring spaces.

Kahn's $\cup_i$-products on the stable stems are consequences of the E_∞ ring structure on QS^0, but their definition uses only a very small part of the total structure available. I suspect that this problem is intimately related to the Arf invariant question in the $(2^s - 2)$-stems.

The notion of E_∞ ring space is clearly essential to infinite loop space theory. There are those who feel that the abstract theory cannot be regarded as complete until the structures used are shown to be homotopy invariant.

<u>Problem 18</u>. Is a space of the homotopy type of an E_∞ ring space again an E_∞ ring space?

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