Maximal Functions in Analysis

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Abstract

This will be a self-contained introduction to the theory of maximal functions, which are some of the most important objects in modern harmonic analysis and partial differential equations. We shall consider various generalizations of the Fundamental Theorem of Calculus, and wind up with an elementary introduction to Calderon-Zygmund Theory. One of the basic ingredients of the study of maximal functions is a deep understanding of the geometry of collections of simple sets such as balls or rectangles in Euclidean spaces. We shall investigate this geometry in some detail.

Contents

1 The Fundamental Theorem of Calculus

Theorem 1.1 (The Fundamental Theorem of Calculus). *If f is continuous then $\left(\int_a^x f\,dx\right)' = f$ and $\int_a^b F'\,dx = F(b) - F(a)$.*

Proof.

$$\lim_{h\to 0} \frac{\int_a^{x+h} f\,dx - \int_a^x f\,dx}{h} = \underbrace{\lim_{h\to 0} \frac{\int_x^{x+h} f\,dx}{h}}_{\text{small average of } f} = f \qquad \square$$

Later, we'll generalize the FTC to $\mathbb{R}^n$ ($n > 1$), and to the Lebesgue Integral.

Definition 1.2. The **Dirichlet Funtion** is defined as:

$$D(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Since it is measurable, the Dirichlet funtion has the property that

$$\int_0^1 D\,dx = 1 \cdot \mu(\{D = 1\}) + 0 \cdot \mu(\{D = 0\})$$

where μ is Lebesgue Measure.

2 Lebesgue Measure

Let's recall some basic facts about Lebesgue Measure on $\mathbb{R}$. If we have

$$\bigcup_{k=1}^{\infty} I_k \supseteq S \quad \text{then} \quad \mu(S) \leq \sum_{k=1}^{\infty} \text{length}(I_k)$$

Definition 2.1. The **Lebesgue Measure** of a set S is defined as:

$$\mu(S) = \inf_{\textit{all covers}} \sum_{k=1}^{\infty} \text{length}(I_k)$$

Remark 2.2.

$$\mu\left(\mathbb{Q} \cap [0,1]\right) < \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon \to 0 \qquad \forall\, \varepsilon > 0$$

Remark 2.3.
$$\mu\left(\overline{\mathbb{Q}\cap[0,1]}\right)=1$$

Remark 2.4. For disjoint measurable sets
$$\mu\left(\bigcup_{k=1}^{\infty}E_k\right)=\sum_{k=1}^{\infty}\mu(E_k)$$

Definition 2.5. A Lebesgue **Simple Function** S is a measurable function with finitely many values $v_1,\dots,v_N$.

Remark 2.6. For a simple function S
$$\int_0^1 S\,dx=\sum_{k=1}^{N}v_k\cdot\mu(\{S=v_k\})$$

Definition 2.7 (Lebesgue Integral). Let $f(x)\geq 0$ be a measurable function on [0,1], then
$$\int_0^1 f(x)\,dx\overset{def}{=}\sup_{\substack{S\,simple\\0\leq S\leq f}}\int_0^1 S\,dx$$

Definition 2.8 (L^p spaces). for $1\leq p<\infty$ define:
$$L^1([0,1])\overset{def}{=}\left\{f\,measurable\mid\int_0^1|f|\,dx<\infty\right\}$$
$$L^p([0,1])\overset{def}{=}\left\{f\,measurable\mid\int_0^1|f|^p\,dx<\infty\right\}$$

Recall the Dirichlet Function
$$D(x)=\begin{cases}1 & x\in\mathbb{Q}\\0 & x\notin\mathbb{Q}\end{cases}$$

We have that $\left(\int_0^x D\,dx\right)'=D$ almost everywhere (that is, except on a set of measure 0). We can verify this: if $x\notin\mathbb{Q}$ then $\left(\int_0^x D(x)\,dx\right)'=0=D(x)=0$, so it's false only when $x\in\mathbb{Q}$ (which has measure 0).

3 The Generalized FTC

Theorem 3.1 (The Generalized FTC for the Lebesgue Integral). *Let* $f \in L^1([0,1])$*. Then*

$$\left(\int_0^x f\,dx\right)' = f \qquad \textit{almost everywhere}$$

Definition 3.2. If $f \in L^1([0,1] \times [0,1])$ then

$$\sup_{h>0} \frac{1}{2h}\int_{x-h}^{x+h} f\,dx = \sup_{x\in I} \frac{1}{|I|}\int_I f\,dx = Mf(x)$$

Definition 3.3. If $f \in L^1(\mathbb{R}^n)$ then the **Maximal Function of f** is

$$M(f)(x) \stackrel{def}{=} \sup_{x\in B} \frac{1}{\mu(B)}\int_B |f|\,dx$$

4 The Hardy-Littlewood Maximal Theorem

Theorem 4.1 (Hardy-Littlewood Maximal Theorem). *If* $f \in L^p(\mathbb{R}^n)$

$$\begin{aligned}
&a) \quad p = 1 \qquad \mu\{M(f)(x) > \alpha\} \leq \frac{C}{\alpha}\|f\|_1 \qquad \textit{(a weak-type on } L^1\textit{)}\\
&b) \quad p > 1 \qquad \|M(f)\|_p \leq C_{p,n}\|f\|_p \qquad \textit{(bounded on } L^p\textit{)}
\end{aligned}$$

The first part of this theorem is related to Chebychev's Inequality.

Theorem 4.2 (Chebychev's Inequality). *If* $Q \geq 0$ *then*

$$\mu(\{Q > \alpha\}) \leq \frac{1}{\alpha}\int_{\mathbb{R}^n} Q\,dx$$

Proof. Starting out with a trivial inequality,

$$\begin{aligned}
\alpha \cdot \chi_{\{Q>\alpha\}}(x) &\leq Q(x)\\
\alpha \int_{\mathbb{R}^n} \chi_{\{Q>\alpha\}}\,dx &\leq \int_{\mathbb{R}^n} Q\,dx\\
\alpha \cdot \mu(\{Q > \alpha\}) &\leq \int_{\mathbb{R}^n} Q\,dx\\
\mu(\{Q > \alpha\}) &\leq \frac{1}{\alpha}\int_{\mathbb{R}^n} Q\,dx
\end{aligned}$$

□

5 The Vitali Covering Lemma

Question: can you get a subset of an arbitrary collection of finitely many balls in $\mathbb{R}^n$ that is disjoint and counts a fixed fraction of their total area?

Lemma 5.1 (The Vitali Covering Lemma). *Let $B_1, B_2, \ldots, B_N$ be balls in $\mathbb{R}^n$. Then $\exists$ a sub-collection of balls, $\widetilde{B}_1, \widetilde{B}_2, \ldots, \widetilde{B}_M$, such that:*

$$\begin{aligned} &a) \quad \text{The } \widetilde{B}_k \text{ are pairwise disjoint} \\ &b) \quad \mu\left(\bigcup_{k=1}^{M} \widetilde{B}_k\right) \geq \frac{1}{3^n} \cdot \mu\left(\bigcup_{k=1}^{N} B_k\right) \end{aligned}$$

Proof. First, arrange the balls by decreasing radius. Choose B_1, so that $\widetilde{B}_1 = B_1$, for B_2 choose $\widetilde{B}_2 = B_2$ if and only if it is disjoint from $\widetilde{B}_1 = B_1$. Continue this - so given a ball B_k, choose it for the sub-collection if and only if it is disjoint from all previously chosen balls.

Claim:

$$\bigcup B_k \subseteq \bigcup \left(\widetilde{B}_j\right)_3$$

Where the notation $(\widetilde{B}_j)_3$ means a concentric enlargement of the ball by a factor of 3.

Pf of Claim: Take one B_k and ask: why is $B_k \subseteq \bigcup \left(\widetilde{B}_j\right)_3$? This is easy to see by drawing a picture, and keeping in mind the way we chose the $\{\widetilde{B}_j\}$ above. So:

$$\begin{aligned} \mu\left(\bigcup B_k\right) \leq \mu\left(\bigcup \left(\widetilde{B}_j\right)_3\right) &\leq \sum \mu\left(\left(\widetilde{B}_j\right)_3\right) \\ &= \sum 3^n \cdot \mu\left(\widetilde{B}_j\right) = 3^n \cdot \mu\left(\bigcup \widetilde{B}_j\right) \end{aligned}$$

since the $\{\widetilde{B}_j\}$ were chosen to be disjoint. $\square$

6 Proof of The Hardy-Littlewood Theorem

Theorem 6.1 (Restating Thm 4.1). *If $f \in L^p(\mathbb{R}^n)$*

$$\begin{aligned} &a) \quad p = 1 \qquad \mu\{M(f)(x) > \alpha\} \leq \frac{C}{\alpha}\|f\|_1 \qquad \text{(a weak-type on } L^1\text{)} \\ &b) \quad p > 1 \qquad \|M(f)\|_p \leq C_{p,n}\|f\|_p \qquad \text{(bounded on } L^p\text{)} \end{aligned}$$

Proof. For $p = \infty$, it's easy to check that $\|M(f)\|_\infty \leq 1 \cdot \|f\|_\infty$. Now, we'll use the Vitali Covering Lemma: $\forall x \in E_\alpha = \{Mf > \alpha\}$, choose B_x such that

$$\frac{1}{\mu(B_x)} \int_{B_x} |f|\, dx > \alpha$$

and $\{B_x\}_{x \in E_\alpha}$ covers E_α. Without loss of generality, $\exists\, B_1, B_2, \ldots, B_k$, a sequence of disjoint open balls from the B_x's such that

$$\bigcup_k B_k \supseteq \bigcup_{x \in E_\alpha} B_x \supseteq E_\alpha$$

So we're going to show

$$\mu\left(\{M(f)(x) > \alpha\}\right) \leq \mu\left(\bigcup_{k=1}^{N} B_k\right) \leq \frac{C}{\alpha}\|f\|_1 \qquad \forall N\, (C\, independent\, of\, N)$$

The proof is now fairly straightforward:

$$\begin{aligned} \mu\left(\bigcup_{k=1}^{N} B_k\right) &= \sum_{k=1}^{N} \mu\left(B_k\right) \\ &< \sum_{k=1}^{N} \frac{1}{\alpha} \int_{B_k} |f|\, dx = \frac{1}{\alpha} \int_{\bigcup B_k} |f|\, dx \leq \frac{1}{\alpha}\|f\|_1 \end{aligned}$$

The second step in this string of inequalities comes from the following observation: look where

$$\frac{1}{\mu(B)} \int_B |f|\, dx > \alpha \quad \Rightarrow \quad \mu(B_k) < \frac{1}{\alpha} \int_{B_k} |f|\, dx \leq \frac{1}{\alpha}\|f\|_1$$

This completes the proof of the Hardy-Littlewood Theorem for $p = 1$ (see section 8 for the other part). $\square$

7 The Generalized FTC Revisited

Theorem 7.1 (Generalized FTC in $\mathbb{R}^n$). *If $f \in L^1(\mathbb{R}^n)$ then*

$$\lim_{r \to 0} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} f\, dx \;=\; f \qquad \textit{outside a set of measure 0}$$

Proof. Given any $f \in L^1(\mathbb{R}^n)$, $\exists$ a 'beautiful' function $C \in L^1(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} |f - C|\, dx$ is as small as you wish. Define:

$$A_r(f)(x) = \frac{1}{\mu(B_r(x))} \int_{B_r(x)} f\, dx$$

We want to prove that

$$\lim_{r \to 0} A_r(f)(x) \longrightarrow f(x) \qquad \text{outside a set of measure } 0$$

Note that A_r is additive:

$$A_r(f_1 + f_2)(x) = A_r(f_1)(x) + A_r(f_2)(x)$$

We have $f \in L^1(\mathbb{R}^n)$. Let $f = C + \eta$ (C is the 'beautiful' function above) where $\eta = error = f - C$ and $\int_{\mathbb{R}^n} |\eta|\, dx < \varepsilon$. Since A_r is additive:

$$A_r(f)(x) = A_r(C)(x) + A_r(\eta)(x)$$

Subtracting equations we get:

$$A_r(f)(x) - f(x) = \underbrace{[A_r(C)(x) - C(x)]}_{this \to 0 \ \ \forall\, x} + [A_r(\eta)(x) - \eta(x)]$$

Now we play some tricks with limsups:

$$\begin{aligned}
&\limsup_{r \to 0} |A_r(f)(x) - f(x)| \\
&= \lim_{k \to 0} \sup_{0 < r < k} |A_r(f)(x) - f(x)| \\
&\leq \underbrace{\limsup_{r \to 0} |A_r(C)(x) - C(x)|}_{this \to 0 \ \text{ by usual FTC}} + \limsup_{r \to 0} |A_r(\eta)(x) - \eta(x)| \\
&\leq \sup_{r > 0} |A_r(\eta)(x)| + |\eta(x)| \\
&\leq M(\eta)(x) + |\eta(x)|
\end{aligned}$$

Remember that we want to prove that

$$\lim_{r \to 0} A_r(f)(x) \longrightarrow f(x) \qquad \text{outside a set of measure } 0$$

so lets look at the following set:

$$\begin{aligned}
&\mu\left\{\limsup_{r\to 0}|A_r(f)(x)-f(x)|>\varepsilon\right\} \\
\leq\, &\mu\left\{M(\eta)>\frac{\varepsilon}{2}\right\}+\mu\left\{|\eta(x)|>\frac{\varepsilon}{2}\right\} \\
\leq\, &\frac{C}{\left(\frac{\varepsilon}{2}\right)}\|\eta\|_1+\mu\left\{|\eta(x)|>\frac{\varepsilon}{2}\right\} && \text{(by H-L)} \\
\leq\, &\frac{2C}{\varepsilon}\|\eta\|_1+\frac{2}{\varepsilon}\|\eta\|_1 && \text{(by Chebychev)}
\end{aligned}$$

□

8 Proof of Hardy-Littlewood when $p > 1$

Theorem 8.1 (Restating Thm 4.1). *If* $f \in L^p(\mathbb{R}^n)$

$$\begin{array}{llll}
a) & p=1 & \mu\{M(f)(x)>\alpha\}\leq \dfrac{C}{\alpha}\|f\|_1 & \textit{(a weak-type on } L^1\textit{)} \\
b) & p>1 & \|M(f)\|_p \leq C_{p,n}\|f\|_p & \textit{(bounded on } L^p\textit{)}
\end{array}$$

Proof. We proved part a) in section 6, so now we're going to prove part b), and we're going to show that a) implies b), that is: $\mu\{M(f)(x) > \alpha\} \leq \frac{C}{\alpha}\|f\|_1$ implies that $\|M(f)\|_p \leq C_{p,n}\|f\|_p$ for $p > 1$. Define for $\alpha > 0$ the **distribution function of f**:

$$\lambda_f(\alpha) = \mu\{|f(x)| > \alpha\}$$

Now we're going to play with some integrals and use Fubini's Theorem:

$$\begin{aligned}
\int_{\mathbb{R}^n}|f(x)|^p\,dx &= (\|f\|_p)^p \\
&= p\int_0^\infty \alpha^{p-1}\lambda_f(\alpha)\,d\alpha \\
&= p\int_0^\infty \alpha^{p-1}\,\mu\{|f|>\alpha\}\,d\alpha \\
&= p\int_0^\infty \alpha^{p-1}\int_{\mathbb{R}^n}\chi_{\{|f(x)|>\alpha\}}\,dx\,d\alpha \\
&= \int_{\mathbb{R}^n}\int_0^{|f(x)|} p\,\alpha^{p-1}\,d\alpha\,dx \\
&= \int_{\mathbb{R}^n}|f(x)|^p\,dx
\end{aligned}$$

Now $f \in L^p(\mathbb{R}^n)$, and define $f = f^\alpha + f_\alpha$ where

$$f^\alpha(x) = \begin{cases} f(x) & where\ |f(x| \geq \frac{\alpha}{2} \\ 0 & where\ |f(x| < \frac{\alpha}{2} \end{cases}$$

and

$$f_\alpha(x) = \begin{cases} 0 & where\ |f(x| \geq \frac{\alpha}{2} \\ f(x) & where\ |f(x| < \frac{\alpha}{2} \end{cases}$$

Since the maximum operator is a supremum of averages of values over balls, it is sub-additive:

$$M(f) = M(f^\alpha + f_\alpha) \leq M(f^\alpha) + M(f_\alpha)$$

Now note that $f^\alpha \in L^1(\mathbb{R}^n)$ and $f_\alpha \in L^\infty(\mathbb{R}^n)$. We're going to make some observations about the sets, and then look at their measures:

$$\begin{aligned} \{M(f) > \alpha\} &\subseteq \left\{M(f^\alpha) > \frac{\alpha}{2}\right\} \cup \left\{M(f_\alpha) > \frac{\alpha}{2}\right\} \\ \mu\{M(f) > \alpha\} &\leq \mu\left\{M(f^\alpha) > \frac{\alpha}{2}\right\} + \mu\left\{M(f_\alpha) > \frac{\alpha}{2}\right\} \\ &\leq \frac{C}{\left(\frac{\alpha}{2}\right)}\|f^\alpha\|_1 \qquad\qquad \text{(by part a) of H-L)} \end{aligned}$$

Now we're going to use the distribution function of Mf:

$$\lambda_{Mf}(\alpha) \leq \frac{2C}{\alpha}\int_{\mathbb{R}^n} |f^\alpha(x)|\, dx = \frac{2C}{\alpha}\int_{\{|f(x)|>\frac{\alpha}{2}\}} |f(x)|\, dx$$

Using the same logic as before (and Fubini again):

$$\begin{aligned}
\int_{\mathbb{R}^n} |Mf(x)|^p \, dx &= (\|Mf\|_p)^p \\
&= p \int_0^\infty \alpha^{p-1} \lambda_{Mf}(\alpha) \, d\alpha \\
&\leq p \int_0^\infty \frac{2C}{\alpha} \int_{\{|f(x)| > \frac{\alpha}{2}\}} |f(x)| \, dx \; \alpha^{p-1} \, d\alpha \\
&= 2pC \int_{\mathbb{R}^n} \int_0^{2|f(x)|} \alpha^{p-2} \, d\alpha \; |f(x)| \, dx \\
&= 2pC \int_{\mathbb{R}^n} \frac{(2|f(x)|)^{p-1}}{p-1} \cdot |f(x)| \, dx \\
&= \frac{2^p pC}{p-1} \int_{\mathbb{R}^n} |f(x)|^p \, dx \\
&= C_p (\|f\|_p)^p
\end{aligned}$$

□

9 Some Closing Questions

Do you think the generalized FTC is still true if we replace $B_r(x)$ with rectangles, and state the theorem the same way? If not, when is it true (by placing restrictions on f)? When is it true in $\mathbb{R}^2$? In $\mathbb{R}^n$?