FINITE CATEGORIES

NOTES FOR THE REU BY J.P. MAY

Since several people have asked for notes, notes shall be provided. To start with, here are my notes to myself for the first day.

July 18

(1) Define categories: objects, morphisms, identity morphisms, composition; associative and unital.
(2) Define isomorphisms of objects in a category
(3) Groups and groupoids. Group: a one object category with every morphism an isomorphism. Groupoid: a many object category with every morphism an isomorphism.
(4) Monoids and categories. Monoid: a one object category. Category: a many object monoid.
(5) Examples, large. Sets, spaces, groups, abelian groups, etc. Contrasted with small categories such as a monoid or a group.
(6) Poset: a category with at most one morphism between any two objects.
(7) Pictures of categories, spaces associated to a category.
(8) Connected categories and disjoint unions. Discrete categories.
(9) Cardinality of a finite connected category: the number of non-identity morphisms, or the number of morphisms minus the number of objects.
(10) Problem: classify finite categories. Under what equivalence relation?
(11) Define functors (covariant only). Define isomorphism of category.
(12) Define natural transformation. Define equivalence of category. Illustrate naturality with $V \mapsto DDV$ for vector spaces over a field K, $DV = \mathrm{Hom}(V, K)$.
(13) Three choices of equivalence: isomorphism, equivalence, homotopy equivalence of associated spaces.

July 19, 20, 21

I don't seem to have notes, but some of what I did was review, examples, and informal introduction of the space associated to a (small) category and the notion of adjoint functors on July 20. The most interesting thing was a pair of adjoint functors relating the category $\mathscr{M}on$ of monoids, the category $\mathscr{M}on_0$ of monoids with zero, and the category $\mathscr{C}at$ of small categories. I will try to find time to write this up, or prevail on somebody else to do so. For the other two days, notes have been written and will be posted, thanks to Niles Johnson and Wei Ren.

You may object that there is no content above, and no full definitions. Ok. Using the wonders of tex, I will shamelessly crib from a published source, although one that is notoriously concise. Here we go.

1. Categories

Category theory gives us a language to describe maps between one area of mathematics and another. We record the basic terminology.

A category $\mathscr{C}$ consists of a collection of objects, a set $\mathscr{C}(A,B)$ of morphisms (also called maps) between any two objects, an identity morphism $\mathrm{id}_A \in \mathscr{C}(A,A)$ for each object A (usually abbreviated id), and a composition law

$$\circ : \mathscr{C}(B,C) \times \mathscr{C}(A,B) \longrightarrow \mathscr{C}(A,C)$$

for each triple of objects A, B, C. Composition must be associative, and identity morphisms must behave as their names dictate:

$$h \circ (g \circ f) = (h \circ g) \circ f, \quad \mathrm{id} \circ f = f, \quad \text{and} \quad f \circ \mathrm{id} = f$$

whenever the specified composites are defined. A category is "small" if it has a set of objects.

We have the category $\mathscr{S}$ of sets and functions, the category $\mathscr{U}$ of topological spaces and continuous functions, the category $\mathscr{G}$ of groups and homomorphisms, the category $\mathscr{A}b$ of Abelian groups and homomorphisms, and so on.

2. Functors

A functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ is a map of categories. It assigns an object $F(A)$ of $\mathscr{D}$ to each object A of $\mathscr{C}$ and a morphism $F(f) : F(A) \longrightarrow F(B)$ of $\mathscr{D}$ to each morphism $f : A \longrightarrow B$ of $\mathscr{C}$ in such a way that

$$F(\mathrm{id}_A) = \mathrm{id}_{F(A)} \quad \text{and} \quad F(g \circ f) = F(g) \circ F(f).$$

More precisely, this is a covariant functor. A contravariant functor F reverses the direction of arrows, so that F sends $f : A \longrightarrow B$ to $F(f) : F(B) \longrightarrow F(A)$ and satisfies $F(g \circ f) = F(f) \circ F(g)$. A category $\mathscr{C}$ has an opposite category $\mathscr{C}^{op}$ with the same objects and with $\mathscr{C}^{op}(A,B) = \mathscr{C}(B,A)$. A contravariant functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ is just a covariant functor $\mathscr{C}^{op} \longrightarrow \mathscr{D}$.

For example, we have forgetful functors from spaces to sets and from Abelian groups to sets, and we have the free Abelian group functor from sets to Abelian groups.

3. Natural transformations

A natural transformation $\alpha : F \longrightarrow G$ between functors $\mathscr{C} \longrightarrow \mathscr{D}$ is a map of functors. It consists of a morphism $\alpha_A : F(A) \longrightarrow G(A)$ for each object A of $\mathscr{C}$ such that the following diagram commutes for each morphism $f : A \longrightarrow B$ of $\mathscr{C}$:

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ {\scriptstyle \alpha_A}\downarrow & & \downarrow{\scriptstyle \alpha_B} \\ G(A) & \xrightarrow[G(f)]{} & G(B). \end{array}$$

Intuitively, the maps α_A are defined in the same way for every A.

For example, if $F : \mathscr{S} \longrightarrow \mathscr{A}b$ is the functor that sends a set to the free Abelian group that it generates and $U : \mathscr{A}b \longrightarrow \mathscr{S}$ is the forgetful functor that sends an Abelian group to its underlying set, then we have a natural inclusion of sets

$S \longrightarrow UF(S)$. The functors F and U are left adjoint and right adjoint to each other, in the sense that we have a natural isomorphism

$$\mathscr{A}b(F(S), A) \cong \mathscr{S}(S, U(A))$$

for a set S and an Abelian group A. This just expresses the "universal property" of free objects: a map of sets $S \longrightarrow U(A)$ extends uniquely to a homomorphism of groups $F(S) \longrightarrow A$.

Formally, functors $L\colon \mathscr{C} \longrightarrow \mathscr{D}$ and $R\colon \mathscr{D} \longrightarrow \mathscr{C}$ are left and right adjoint if there is a natural isomorphism

$$\mathscr{D}(LA, B) \cong \mathscr{C}(A, RB).$$

Two categories $\mathscr{C}$ and $\mathscr{D}$ are equivalent if there are functors $F : \mathscr{C} \longrightarrow \mathscr{D}$ and $G : \mathscr{D} \longrightarrow \mathscr{C}$ and natural isomorphisms $FG \longrightarrow \mathrm{Id}$ and $GF \longrightarrow \mathrm{Id}$, where the Id are the respective identity functors.

4. Homotopy categories and homotopy equivalences

Let $\mathscr{T}$ be the category of spaces X with a chosen basepoint $x \in X$; its morphisms are continuous maps $X \longrightarrow Y$ that carry the basepoint of X to the basepoint of Y. The fundamental group specifies a functor $\mathscr{T} \longrightarrow \mathscr{G}$, where $\mathscr{G}$ is the category of groups and homomorphisms.

When we have a (suitable) relation of homotopy between maps in a category $\mathscr{C}$, we define the homotopy category $h\mathscr{C}$ to be the category with the same objects as $\mathscr{C}$ but with morphisms the homotopy classes of maps. We have the homotopy category $h\mathscr{U}$ of unbased spaces. On $\mathscr{T}$, we require homotopies to map basepoint to basepoint at all times t, and we obtain the homotopy category $h\mathscr{T}$ of based spaces. The fundamental group is a homotopy invariant functor on $\mathscr{T}$, in the sense that it factors through a functor $h\mathscr{T} \longrightarrow \mathscr{G}$.

A homotopy equivalence in $\mathscr{U}$ is an isomorphism in $h\mathscr{U}$. Less mysteriously, a map $f : X \longrightarrow Y$ is a homotopy equivalence if there is a map $g : Y \longrightarrow X$ such that both $g \circ f \simeq \mathrm{id}$ and $f \circ g \simeq \mathrm{id}$. Working in $\mathscr{T}$, we obtain the analogous notion of a based homotopy equivalence. Functors carry isomorphisms to isomorphisms, so we see that a based homotopy equivalence induces an isomorphism of fundamental groups. The same is true, less obviously, for unbased homotopy equivalences.

Proposition. *If $f : X \longrightarrow Y$ is a homotopy equivalence, then*

$$f_* : \pi_1(X, x) \longrightarrow \pi_1(Y, f(x))$$

is an isomorphism for all $x \in X$.

A space X is said to be contractible if it is homotopy equivalent to a point.

Corollary. *The fundamental group of a contractible space is zero.*

5. The fundamental groupoid

While algebraic topologists often concentrate on connected spaces with chosen basepoints, it is valuable to have a way of studying fundamental groups that does not require such choices. For this purpose, we define the "fundamental groupoid" $\Pi(X)$ of a space X to be the category whose objects are the points of X and whose morphisms $x \longrightarrow y$ are the equivalence classes of paths from x to y. Thus the set of endomorphisms of the object x is exactly the fundamental group $\pi_1(X, x)$.

The term "groupoid" is used for a category all morphisms of which are isomorphisms. The idea is that a group may be viewed as a groupoid with a single object. Taking morphisms to be functors, we obtain the category $\mathscr{G}\mathscr{P}$ of groupoids. Then we may view Π as a functor $\mathscr{U} \longrightarrow \mathscr{G}\mathscr{P}$.

There is a useful notion of a skeleton $sk\mathscr{C}$ of a category $\mathscr{C}$. This is a "full" subcategory with one object from each isomorphism class of objects of $\mathscr{C}$, "full" meaning that the morphisms between two objects of $sk\mathscr{C}$ are all of the morphisms between these objects in $\mathscr{C}$. The inclusion functor $J : sk\mathscr{C} \longrightarrow \mathscr{C}$ is an equivalence of categories. An inverse functor $F : \mathscr{C} \longrightarrow sk\mathscr{C}$ is obtained by letting $F(A)$ be the unique object in $sk\mathscr{C}$ that is isomorphic to A, choosing an isomorphism $\alpha_A : A \longrightarrow F(A)$, and defining $F(f) = \alpha_B \circ f \circ \alpha_A^{-1} : F(A) \longrightarrow F(B)$ for a morphism $f : A \longrightarrow B$ in $\mathscr{C}$. We choose α to be the identity morphism if A is in $sk\mathscr{C}$, and then $FJ = \mathrm{Id}$; the α_A specify a natural isomorphism $\alpha : \mathrm{Id} \longrightarrow JF$.

A category $\mathscr{C}$ is said to be connected if any two of its objects can be connected by a sequence of morphisms. For example, a sequence $A \longleftarrow B \longrightarrow C$ connects A to C, although there need be no morphism $A \longrightarrow C$. However, a groupoid $\mathscr{C}$ is connected if and only if any two of its objects are isomorphic. The group of endomorphisms of any object C is then a skeleton of $\mathscr{C}$. Therefore the previous paragraph specializes to give the following relationship between the fundamental group and the fundamental groupoid of a path connected space X.

Proposition. *Let X be a path connected space. For each point $x \in X$, the inclusion $\pi_1(X, x) \longrightarrow \Pi(X)$ is an equivalence of categories.*

Proof. We are regarding $\pi_1(X, x)$ as a category with a single object x, and it is a skeleton of $\Pi(X)$. □