FINITE CATEGORIES: JULY 19

NOTES FOR THE REU

Last time we defined the notions of isomorphism and equivalence of categories. To clarify these notions, we give an example of two categories which are equivalent, but not isomorphic.

Example 0.1. Let $\mathscr{C}$ be a category with one object, X, and one non-identity morphism $g : X \to X$, with $g \circ g = 1_X$. Let $\mathscr{D}$ be a category with two objects, Y and Z, with four non-identity morphisms:

- $h : Y \to Y$, with $h \circ h = 1_Y$
- $j : Z \to Z$, with $j \circ j = 1_Z$
- $f : Y \to Z$
- $f^{-1} : Z \to Y$, with $f \circ f^{-1} = 1_Z$ and $f^{-1} \circ f = 1_Y$.

Define functors $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ as follows.

$$\begin{array}{ll} F(X) = Y & G(Y) = G(Z) = X \\ F(g) = h & G(h) = G(j) = g \\ & G(f) = G(f^{-1}) = 1_X \end{array}$$

Write down the composites FG and GF explicitly, and find a natural transformation $\alpha : \mathrm{Id} \to FG$.

Note 0.2. The functor F is not unique. Give an example of another functor $\mathscr{C} \to \mathscr{D}$ which will also be an equivalence of categories.

Exercise 0.3. A monoid is a category with one object. The elements of the monoid are the morphisms in the category (all from the one object to itself), and multiplication in the monoid is the rule for composition in the category.

Problem 0.4. *A preliminary problem to the problem of classifying categories with n (non-identity) morphisms could be the problem of classifying monoids with n (non-identity) morphisms. Since there is only one object in a monoid, there is only one identity morphism, and hence $n + 1$ morphisms all together.*

Exercise 0.5. There is only one group of order two, but there are two monoids of order two. Their multiplication tables are.

	1	g
1	1	g
g	g	1

	1	g
1	1	g
g	g	g

which one is the group?

The preceeding example gives us also an example of a monoid with a zero element.

Definition 0.6. A *zero element* in a monoid M is an element $z \in M$ such that $zx = z = xz$ for all $x \in M$.

Problem 0.7. *What are all the monoids of order three? Write down all possible multiplication tables for three elements, $1, g, h$. Make sure your multiplication is associative!*