FINITE CATEGORIES: JULY 21

NOTES FOR THE REU

Recall the category $\mathscr{P}$ defined previously: $\mathscr{P}$ has one object, denoted $*$ and one morphism (the morphism is the identity morphism, 1_* on the one object). For any category $\mathscr{C}$, and any object $X \in \mathscr{C}$, there is a functor $\iota_X : \mathscr{P} \to \mathscr{C}$. The functor ι_X is defined on the one object of $\mathscr{P}$ by $\iota_X * = X$ and on the one morphism of $\mathscr{P}$ by $\iota(1_*) = 1_X$.

Note, furthermore, that there is only one functor $\mathscr{C} \to \mathscr{P}$ (what is it?). Let p denote this functor.

Definition 0.1. We say X is an *initial object* of $\mathscr{C}$ if for all $Y \in \mathscr{C}$ there is exactly one morphism $j_Y : X \to Y$ in $\mathscr{C}$. That is to say, the set $\mathscr{C}(X, Y)$ has one element for all $Y \in \mathscr{C}$.

Proposition 0.2. *The object X is initial iff the functor $\iota_X : \mathscr{P} \to \mathscr{C}$ is left-adjoint to $p : \mathscr{C} \to \mathscr{P}$.*

Exercise 0.3. give this proof

Definition 0.4. We say X is a *terminal object* of $\mathscr{C}$ if for all $Y \in \mathscr{C}$ there is exactly one morphism $q_Y : Y \to X$ in $\mathscr{C}$. That is to say, the set $\mathscr{C}(Y, X)$ has one element for all $Y \in \mathscr{C}$.

Exercise 0.5. Show that X is terminal iff ι_X is right-adjoint to p.

Exercise 0.6. Now recall the example given previously of the category $\mathscr{P}$ together with the category $\mathscr{I}$. The category $\mathscr{I}$ has two objects, [0] and [1], and one non-identity morphism, $[\mathrm{I}] : [0] \to [1]$. We defined two functors $\mathscr{P} \to \mathscr{I}$, and using the notions above, you should now be able to find which of these is left-adjoint to p and which is right-adjoint to p.

Example 0.7. Let $\mathscr{D}$ be a category with three objects, and two non-identity morphisms, and let $\mathscr{C}$ be a category with two objects and one non-identity morphism, as in the diagram below.

$$b \xleftarrow{f} a \xrightarrow{g} c \qquad \mathscr{D}$$

$$b \xleftarrow{f} a \qquad\qquad \mathscr{C}$$

There is no pair of adjoint functors between $\mathscr{C}$ and $\mathscr{D}$ (check this). Is there some intermediate category (or categories) such that there is a string of adjoint pairs connecting $\mathscr{C}$ to $\mathscr{D}$ through the intermediate category?

Note 0.8. In understanding this example, it may be helpful to observe how adjoint functors treat terminal and initial objects. If F and G are an adjoint pair of functors (with F left-adjoint to G), and X is an initial object, must $F(X)$ be? Is the converse true? What if X is terminal?

0.1. **Contravariant Functors.** Recall that the definition of functor requires that, for a morphism $f : X \to Y$, $F(f)$ is a morphism $F(X) \to F(Y)$. There are, however, "functors" G which satisfy all the other properties of a functor, but instead of giving morphisms $G(X) \to G(Y)$, give morphisms going the other direction, $G(Y) \to G(X)$. Some people call these new "functors" *contravariant functors*, and then use the word *covariant functor* for functors in the original sense. We can, however, fit contravariant functors into the framework we've already developed by introducing the important (but simple) notion of an opposite category.

Definition 0.9. If $\mathscr{C}$ is a category, we form the *opposite category* of $\mathscr{C}$, denoted $\mathscr{C}^{op}$, by reversing the directions of the arrows in $\mathscr{C}$.

To say that in a more precise way: The objects of $\mathscr{C}^{op}$ are the same as the objects of $\mathscr{C}$, and for two objects $X, Y \in \mathscr{C}^{op}$, $\mathscr{C}^{op}(X, Y)$ is defined to be the set $\mathscr{C}(Y, X)$.

Example 0.10. The opposites for the categories $\mathscr{C}$ and $\mathscr{D}$ of example 0.7 are:

$$b \xrightarrow{f^{op}} a \xleftarrow{g^{op}} c \qquad \mathscr{D}^{op}$$

$$b \xrightarrow{f^{op}} a \qquad\qquad \mathscr{C}^{op}$$

Now if $G : \mathscr{C}^{op} \to \mathscr{D}$ is a functor, what does this mean in terms of the category $\mathscr{C}$? If X is an object of $\mathscr{C}$, then it is also an object of $\mathscr{C}^{op}$, and so $G(X)$ is an object of $\mathscr{D}$. If $f : X \to Y$ is a morphism in $\mathscr{C}$, then the morphism in $\mathscr{C}^{op}$ is $f^{op} : Y \to X$ and hence $G(f^{op}) : G(Y) \to G(X)$. We can call G a contravariant functor $\mathscr{C} \to \mathscr{D}$, or we can call G a (covariant) functor $\mathscr{C}^{op} \to \mathscr{D}$ in the sense of our original definition; there is no difference in the concepts.

Example 0.11. Let $\mathscr{C}$ be the category of finite dimensional vector spaces over $\mathbb{R}$ and define a functor $D : \mathscr{C}^{op} \to \mathscr{S}$ in the following way, where $\mathscr{S}$ is the category of sets and functions between sets. For each finite dimensional vector space over $\mathbb{R}$ (object of $\mathscr{C}$), V, let $D(V)$ be the set of linear transformations $V \to \mathbb{R}$. That is (in more precise notation):

$$\text{for each } V \in \mathscr{C},\ D(V) = \mathscr{C}(V, \mathbb{R}) \in \mathscr{S}$$

For a morphism (linear transformation) $T : V \to W$, $D(T^{op})$ is the morphism $D(W) \to D(V)$ given by precomposing with T, so for an element

$$f \in D(W) = \mathscr{C}(W, \mathbb{R}) = \{\text{linear transformations } W \to \mathbb{R}\},$$
$$D(T)(f) = f \circ T \in D(V) = \mathscr{C}(V, \mathbb{R}) = \{\text{linear transformations } V \to \mathbb{R}\}.$$

Exercise 0.12. Show that D is a functor $\mathscr{C}^{op} \to \mathscr{S}$

Note 0.13. The construction above did not depend on the fact that $\mathscr{C}$ happened to be the category of finite dimensional vector spaces over $\mathbb{R}$. A similar definition could have been made for any category $\mathscr{C}$ and any object $K \in \mathscr{C}$.

$$R_K : \mathscr{C} \to \mathscr{S}$$

$$\text{f}or\ Y \in \mathscr{C},\ R_K(Y) = \mathscr{C}(Y, K).$$

Functors so defined are called *represented* functors.