FINITE CATEGORIES: JULY 25

NOTES FOR THE REU

0.1. **Monoids and Adjoint functors.** We wish to find all monoids with 3 elements. We observe that we can add an identity to any semi-group with 2 elements and obtain a monoid with 3 elements. Surprisingly, even with such a small semigroup, we have at least one semigroup that is not a monoid.

Consider the semigroup $\{g, h\}$, where the multiplication is defined as follows: $gh = h, hg = g, g^2 = g, h^2 = h$. Neither g or h is a two-sided identity.

Exercise 0.1. Determine all the semigroups of size 2. Find all monoids of size 3.

Before we move onto something entirely different, we restate a fact about adjoint functor pairs, which we used implicitly last time.

Proposition 0.2. *Suppose $\mathscr{C}, \mathscr{D}, \mathscr{E}$ are categories, with the functors,*

$$\mathscr{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathscr{D} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{E} .$$

Further assume that L, R are adjoints, and so are F, G. Then the composites, $F \circ L$ and $R \circ G$ are adjoint functors as well.

Proof: By definition of adjoints, for any $X \in \mathscr{O}b(\mathscr{C}), Z \in \mathscr{O}b(\mathscr{E})$, we have the following natural isomorphisms:

$$\mathscr{E}(FLX, Z) \cong \mathscr{D}(LX, GZ) \cong \mathscr{C}(X, RGZ).$$

$\square$

0.2. **Simplicial Complex.** Before we define simplicial sets and the nerve of a category, we first look at the simplicial complex.

Consider the Euclidean space $\mathbf{R}^N$, for some large N. We pick $n+1$ "geometrically independent" points $\{v_0, \ldots, v_n\}$, in the sense that the vectors $\{v_1 - v_0, \ldots, v_n - v_0\}$ are linearly independent.

Definitions 0.3. The set

$$\Delta_n := \{\sum_{\imath=0}^{n} t_i v_i | 0 \le t_i \le 1 \ \sum_{\imath=0}^{n} t_i = 1\}$$

is an $n-$*simplex*. The points v_i's are the vertices, and the t_i's are the coordinates. For a given $n-$simplex, the subset

$$\{\sum_{\imath=0}^{n} t_i v_i | 0 \le t_i \le 1,\ t_k = 0,\ \sum_{\imath=0}^{n} t_i = 1\}$$

is the k^{th} *face* of the simplex.

Note that a $1-$simplex is a line segment, a $2-$simplex is a (filled-in) triangle, a $3-$simplex is a tetrahedron and so on. Also notice that a face of a $n-$simplex is an $(n-1)-$simplex. We will refer to a point in a simplex by its coordinates, $(t_0, \ldots, t_n)$. Having defined a simplex, we can now define:

Definition 0.4. A geometric simplicial complex K is a set of simplices in some $\mathbf{R}^N$ such that each face of a simplex in K is again a simplex in K, and the intersection two simplices of K is another simplex in K. The set of vertices of K is simply the union of all vertices of all of its simplices.

There are some natural operations we can define that will allow us to compare simplices with simplices of higher or lower dimension. We define a map $\delta_i : \Delta_{n-1} \to \Delta_n, 0 \leq i \leq n$ by

$$\delta_i(t_0, \ldots, t_{n-1}) = (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1}).$$

We define a map $\sigma_i : \Delta_{n+1} \to \Delta_n, 0 \leq i \leq n$, by

$$\sigma_i(t_0, \ldots, t_{n+1}) = (t_0, \ldots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \ldots, t_{n+1}).$$

A map between two geometric simplicial complexes is a map that sends vertices to vertices and is linear on simplices. We can thus form the category of geometric simplicial complexes. All of the information about a geometric simplicial is captured in its vertices, and this leads us to the following definition.

Definition 0.5. An *abstract simplicial complex* K is a set $V = V(K)$ of vertices, together with a set K of non-empty subsets of V, called *simplices*, such that

(i) every vertex is in some simplex;
(ii) every subset of a simplex is also a simplex.

The abstract simplicial complexes form a category, where a map between two such complexes is a map between the vertex sets which sends simplices to simplices. A geometric simplicial complex can be seen as an abstract simplicial complex, if we take V to be its set of vertices, and define K according to the simplices in the complex. This operation can be seen as a functor from the category of geometric simplicial complexes to the category of abstract simplicial complexes.

In the other direction, given an abstract simplicial complex, we can define its geometric realization in some $\mathbf{R}^N$, if we choose a bijection between a set of geometrically independent points in $\mathbf{R}^N$ and the vertex set V.

0.3. **Categories and Simplicial Sets.** Now we turn our attention to a more general construction, known as the simplicial set. This will eventually allow us to construct a topological space from a category.

Suppose $\mathscr{C}$ is a category. For any $n \in \mathbf{N}$, we might be able to find a string of $n-$composable maps, $\{f_n, \ldots, f_1\}$, where the successive compositions $f_i \circ f_{i-1}$, $2 \leq i \leq n$, make sense:

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_{n-1} \xrightarrow{f_n} X_n \ .$$

Now we can define $C_n := \{(f_n, \ldots, f_1)\}$ to be the set of all such n-tuples of composable maps. We define $C_0 = \mathscr{O}b(\mathscr{C})$.

We now define maps $d_i : C_n \to C_{n-1}$:

$$d_0(f_n, \dots, f_1) = (f_n, \dots, f_2);$$
$$d_i(f_n, \dots, f_1) = (f_n, \dots, f_{i+2}, f_{i+1} \circ f_i, f_{i-1}, \dots, f_1),\ 1 \leq i \leq n-1;$$
$$d_n(f_n, \dots, f_1) = (f_{n-1}, \dots, f_1).$$

We also define $s_i : C_n \to C_{n+1}, 0 \leq i \leq n$:

$$s_i(f_n, \dots, f_1) = (f_n, \dots, f_{i+1}, \mathrm{Id}, f_i, \dots, f_1), \quad \text{where Id is the appropriate identity map..}$$

Thus we obtain a sequence of sets C_n, within maps s_i and d_i.

Similarly, for any abstract simplicial complex K, we can define

$$K_n := \{ \text{ simplices with } n+1 \text{ vertices in } K\}.$$

We have $K_0 = V$, the vertex set. To define maps between K_n, we first fix an ordering on V. Note that now all subsets of V are totally ordered. Define $d_i : K_n \to K_{n-1}, 0 \leq i \leq n$ to be the map that deletes the i^{th} vertex from each simplex in K_n. Define $s_i : K_n \to K_{n+1}$ to be the map that repeats the i^{th} vertex of each simplex. (Note that for this definition to make sense, we must allow our sets to have repeated elements.)

We have thus far illustrated two common examples of simplicial sets. We have the following definition, to be completed next time:

Definition 0.6. A *simplicial set* is a sequence of sets K_n (indexed by $\mathbf{N}$), with maps $d_i : K_n \to K_{n-1}$, $s_i : K_n \to K_{n+1}$, which satisfy the following commutation relations: