FINITE CATEGORIES: JULY 26

NOTES FOR THE REU

In the previous lecture, abstract simplicial complexes were introduced. Since they are essential to this lecture as well, the definition is repeated.

Definition 1.1. An *abstract simplicial complex* K is a set of vertices $V(K)$ and a set of nonempty subsets of $V(K)$, called simplices, such that

(1) Every vertex is in some simplex.
(2) Every subset of a simplex is a simplex..

1.1. **Background on Topological Spaces.**

An earlier handout gives an overview of topological spaces, but a few concepts are reiterated here for review.

The spaces about which we have the most intuition, and therefore think about the most, are Euclidean spaces $\mathbb{R}^n$ and subsets of Euclidean spaces. The Euclidean space $\mathbb{R}^n$ has a distance function

$$d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$
$$d(x, y) = \sqrt{\sum(x_i - y_i)^2}.$$

Then define $U_{x,\epsilon}$, the ϵ-ball centered at x, to be $\{y \in \mathbb{R}^n : d(x, y) < \epsilon\}$. The open sets of $\mathbb{R}^n$ are all the finite intersections and arbitrary unions of these sets for any choice of x and ϵ.

The concept of a topological space generalizes Euclidean spaces:

Definition 1.2. A *topological space* is a set X and a set $\mathscr{U}$ of subsets of X, called the open sets of X, such that $\emptyset$ and X are in $\mathscr{U}$ and finite intersections and arbitrary unions of elements of $\mathscr{U}$ are in $\mathscr{U}$.

This definition is much more general than Euclidean space. For example, an arbitrary topological space does not necessarily satisfy the following familiar separation property of $\mathbb{R}^n$: for all distinct $x, y \in X$, there exist open sets U and V such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.

A set function $f : X \to Y$ from one topological space X to another Y may not take nearby points in X to nearby points in Y, so we need a restriction on the types of maps between topological spaces. These maps are called continuous, and in the case of metric spaces they are the ones which satisfy the usual ϵ-δ definition: f is continous if for all $x \in X$ and all $\epsilon > 0$, there is a $\delta > 0$ such that $f(U_{x,\delta}) \subseteq U_{f(x),\epsilon}$. The generalized definition for arbitrary topological spaces, which is equivalent to the ϵ-δ definition in the case of metric spaces, is

Definition 1.3. A function $f : X \to Y$ from a topological space X to a topological space Y is *continuous* if $f^{-1}(V)$ is open in X for all open V in Y.

1.2. **Simplicial Sets.**

Even though a topological space is essentially a geometric object, we can study them by assigning algebraic objects to them. Our next step is to develop the tools to investigate topological spaces algebraically.

Suppose we totally order the set of vertices $[n]$ so that it makes sense to refer to the ith vertex. Let K_n be the set of $n+1$ point simplices allowing redundancies. Define functions for $0 \leq i \leq n$

$$\begin{array}{ll} d_i : K_n \to K_{n-1} & \text{delete } i\text{th entry} \\ s_i : K_n \to K_{n+1} & \text{repeat } i\text{th entry.} \end{array}$$

Recall that a category $\mathscr{C}$ defines a sequence of sets and functions by $\mathscr{C}_0 = \mathrm{Obj}(\mathscr{C})$ and $\mathscr{C}_n = \{(f_n, \ldots, f_1) \,|\, \text{the } f_i \text{ are composable}\}$. Then define

$$\begin{array}{l} d_0(f_n, \ldots, f_1) = (f_n, \ldots, f_2) \\ d_i(f_n, \ldots, f_1) = (f_n, \ldots, f_{i+1} \circ f_i, \ldots, f_1) \\ d_n(f_n, \ldots, f_1) = (f_{n-1}, \ldots, f_1) \\ s_i(f_n, \ldots, f_1) = (f_n, \ldots, f_{i+1}, id, f_i, \ldots, f_1) \end{array}$$

Note that in both cases, these functions satisfy the following relations:

$$\begin{array}{ll} d_i d_j & = d_{j-1} d_i \ \ i < j \\ s_i s_j & = s_{j+1} s_i \ \ i \leq j \\ d_i s_j & = \left\{ \begin{array}{ll} s_{j-1} d_i & i < j \\ id & i = j, j+1 \\ s_j d_{i-1} & i > j+1. \end{array} \right. \end{array}$$

Define a category $\mathbb{\Delta}$ whose objects are the sets $\underline{n} = \{0, 1, \ldots, n\}$ and whose morphisms $\tau : \underline{m} \to \underline{n}$ are monotonic nondecreasing functions. That is, if $i < j$, then $\tau(i) \leq \tau(j)$. For example, there are morphisms $\delta_i : \underline{n} \to \underline{n+1}$ defined by

$$\delta_i(j) = \left\{ \begin{array}{ll} j & j < i \\ j+1 & j \geq i. \end{array} \right.$$

The δ_i increment every element greater than or equal to i by one. There are also morphisms $\sigma_i : \underline{n+1} \to \underline{n}$ defined by

$$\sigma_i(j) = \left\{ \begin{array}{ll} j & j \leq i \\ j-1 & j > i. \end{array} \right.$$

Exercise 1.4. Every morphism in $\mathbb{\Delta}$ is a composite of δ_is and σ_is.

In these terms, *a simplicial set is a contravariant functor from* $\mathbb{\Delta}$ *to Sets.* Note the contravariance switches the relations around. Moreover, by the exercise, one need not define the functor on all morphisms of $\mathbb{\Delta}$, but only on the δ_i and σ_i, which we have already done for abstract simplicial complexes and for categories. That is, we have already defined a functor from abstract simpicial complexes to simplicial sets, $K \to K_*$, and from (small) categories to simplicial sets, $\mathscr{C} \to \mathscr{C}_*$. Now we would like to define a functor from topological spaces to simplicial sets, $X \to S_*X$.

Let $S_n(X)$ be the set of continous functions $\Delta_n \to X$. If $\tau : \underline{m} \to \underline{n}$ is a morphism in $\mathbb{\Delta}$, then τ also determines a continuous map $\Delta_m \to \Delta_n$. Thus for each continous $f : \Delta_n \to X$, define $t(f) = f \circ \tau : \Delta_m \to X$, which is also a continous map. Then t maps $S_n(X)$ to $S_m(X)$. Next time we will see how the sets K_*, $\mathscr{C}_*$, or $S_*(X)$ lead to a sequence of abelian groups, which will be algebraic objects associated to the simplicial complex K, category $\mathscr{C}$, or topological space X.