FINITE CATEGORIES

NOTES FOR THE REU: JULY 27
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1. Simplicial Spaces

Informally, we can think of simplicial spaces as spaces built out of oriented lines, triangles, tetrahedra...

2. Homology of simplicial complexes.

Let K_n be the set of $n+1$ point simplices. Recall that the free abelian group $\mathbb{Z}[K_n]$ consists of formal sums of elements of K_n (i.e. $\sum_{i=1}^{n} k_i\sigma_i \in \mathbb{Z}[K_n]$ where $\sigma_i \in K_n$ and $k_i \in \mathbb{Z}$ for each i). We also recall that we were given maps $d_i : K_n \to K_{n-1}$ and $s_i : K_n \to K_{n+1}$ for $i = 0 \dots n$, that satisfy certain relations. Using these maps we define $d : \mathbb{Z}[K_n] \to \mathbb{Z}[K_{n-1}]$ as $d(\sigma) = \sum_{i=0}^{n}(-1)^i d_i(\sigma)$.

We compute

$$
\begin{aligned}
(dd)(\sigma) &= d(\sum_{i=0}^{n}(-1)^i d_i(\sigma)) \\
&= \sum_{0\leq j<n}\sum_{0\leq i\leq n}(-1^{i+j})d_j d_i(\sigma) \\
&= \sum_{0\leq j<i\leq n}(-1)^{i+j}d_j d_i(\sigma) + \sum_{0\leq i\leq j<n}(-1)^{i+j}d_j d_i(\sigma) \\
&= -\sum_{0\leq j<i\leq n}(-1)^{i+j}d_{i-1}d_j(\sigma) + \sum_{0\leq i\leq j<n}(-1)^{i+j}d_j d_i(\sigma) \\
&= -\sum_{0\leq j\leq i<n}(-1)^{i+j}d_i d_j(\sigma) + \sum_{0\leq i\leq j<n}(-1)^{i+j}d_j d_i(\sigma) \\
&= 0
\end{aligned}
$$

Using this we define $B_n(K) = d(\mathbb{Z}[K_{n+1}])$ and $Z_n(K) = \{\sigma \in \mathbb{Z}[K_n] \,|\, d(\sigma) = 0\}$. The *nth* homology group is defined to be $H_n(K) = Z_n(K)/B_n(K)$.

Example 2.1. If we let K be the triangle with ordered points $a < b < c$ and edges f, connecting a to b, g connecting b to c and h connecting a to c. We can compute $H_0(K) = H_1(K) = \mathbb{Z}$. Since $d(\mathbb{Z}[K_0])$ is by definition 0 we see that $Z_0(K) = \mathbb{Z}^3 = \mathbb{Z}a \oplus \mathbb{Z}b \oplus \mathbb{Z}c$. We have that $d(f) = b - a$, $d(g) = c - b$ and $d(h) = c - a = d(f) + d(h)$. So $B_0(K) = d(\mathbb{Z}[K_1]) = \mathbb{Z}(b-a) \oplus \mathbb{Z}(c-b)$. Rewriting $Z_0(K)$ as $\mathbb{Z}(b-a) \oplus \mathbb{Z}(c-b) \oplus \mathbb{Z}b$ we see that $H_0(K) = \mathbb{Z}$. Since there are no 2-simplices we have $B_1(K) = 0$ which implies $H_1(K) = Z_1(K) = \mathbb{Z}(h - (f+g)) = \mathbb{Z}$.

3. The functoriality of homology.

Suppose we have a map of chain complexes $f : C_* \to C'_*$. That is we have that $df_n = f_n d$. We see that if $x \in Z_n(C)$ then $df_n(x) = f_n(dx) = f_n(0) = 0$ so $f(x) \in Z_n(C')$. And we see that if $x \in B_n(C)$, so $x = dy$ for some $y \in C_{n+1}$, then $f(x) = f(dy) = d(f(y))$ and we have $f(x) \in B_n(C')$. These two facts imply that we have a well defined map $H_n(C) \to H_n(C')$.

4. The homotopy invariance of homology

We return to one of our favorite simplicial complexes I. Recall I has two 0-simplices, $I_0 = \{[0],[1]\}$, and one 1-simplex, $I_1 = \{[I]\}$. Let K be a simplicial space, we can check that $\mathbb{Z}[(K \times I)_n] = \mathbb{Z}[K_n \times I_n] = \mathbb{Z}[K_n] \otimes \mathbb{Z}[I_n]$. The n-chains $C_n(K \times I)$ then are given by $\mathbb{Z}[K_n] \otimes [0] \oplus \mathbb{Z}[K_n] \otimes [1] \oplus \mathbb{Z}[K_{n-1}] \otimes [I]$. For $x \in C_n(K)$ we set

$$\begin{aligned} d(x \otimes [0]) &= d(x) \otimes [0] \\ d(x \otimes [1]) &= d(x) \otimes [1] \\ d(x \otimes [I]) &= d(x) \otimes [I] + (-1)^n x \otimes [1] - (-1)^n x \otimes [0]. \end{aligned}$$

Now we see two chain maps $f, g : C \to C'$ are homotopic if there exists a map $h : C \otimes I \to C'$ such that $h(x \otimes [0]) = f(x \otimes [0])$ and $h(x \otimes [1]) = g(x \otimes [1])$ for all $x \in C$.

Theorem 4.1. *If f and g are homotopic then $f_* = g_*$.*