FINITE CATEGORIES

NOTES FOR THE REU: JULY 28
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1. Simplicial Sets and Chain Complexes

Recall that if we have a chain complex

$$\cdots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \longrightarrow \cdots,$$

we define the *homology* groups to be

$$H_n = \frac{Z_n}{B_n},$$

where the $Z_n := \ker(d_n)$ consists of *cycles*, and $B_n := \operatorname{Im}(d_{n+1})$ is the group of *boundaries*. The condition $d^2 = 0$, i.e. $d_n \circ d_{n+1} = 0$, guarantees $B_n \subset Z_n$ and this is well-defined.

Now suppose we have a simplicial set, $\{K_n\}$, with maps d_i and s_i. We define $A_n(K)$ to be the free abelian group generated by K_n. In other words, the elements of $A_n(K)$ are finite formal $\mathbb{Z}-$linear combinations of elements of K_n:

$$A_n(K) = \{ \sum_{k_i \in K_n} n_i k_i, n_i \in \mathbb{Z}\}.$$

The maps $d_i : K_n \to K_{n-1}$ naturally extend to $A_n(K)$ by linearity (that is, we define $d_i(\sum n_i k_i) = \sum n_i d(k_i)$). Now we can define $d : A_n(K) \to A_{n-1}(K)$ by

$$d := \sum_{i=0}^{n} (-1)^i d_i.$$

This map makes the sequence $\{A_n(K)\}$ into a chain complex, because

$$d \circ d = \sum_{i=0}^{n-1} \sum_{j=0}^{n} (-1)^{i+j} d_i \circ d_j,$$

and the identity $d_i \circ d_j = d_{j-1} \circ d_i, i < j$ ensures that $d \circ d = 0$.

We can now define $C_n(A) := A_n(K)/D_n(K)$, where $D_n(K)$ is the degenerate subgroup that consists of all the degenerate elements, which are finite linear combinations of elements of the form $s_i(y)$. Note that commutation relations between d_i and s_i ensure that the map $d : A_n(K) \to A_{n-1}(K)$ descends to the quotient, and we get a map $d : C_n(K) \to C_{n-1}(K)$. This is equivalent to saying that the maps that we obtain by restriction, $d : D_n(K) \to A_{n-1}(K)$, is in fact a map into the degenerate subgroup $D_{n-1}(K)$, and therefore the induced map on the quotient groups $C_n(K) \to C_{n-1}(K)$ is well-defined.

Yet another way of stating this fact is to say that the groups $D_n(K)$ form a subchain complex, with maps d obtained by restriction.

Exercise 1.1. Verify the claims made above. Also check that $H_n(D_*(K))$ is trivial for all n. We will see later that this implies we have an isomorphism: $H_*(A_*(K)) \cong H_*(C_*(K))$.

Now recall that we have defined the singular complex. For any topology space X, S_n is the set of all continuous maps $\{f : \Delta_n \to X\}$, where Δ_n is the standard $n-$simplex. The sets $\{S_n\}$ naturally come with the operators d_i and s_i, by pre-composing with the maps $\delta_i : \Delta_{n-1} \to \Delta_n$ and $\sigma_i : \Delta_{n+1} \to \Delta_n$, which we have defined previously. We define $d_i : S_n \to S_{n-1}$ by $f \mapsto f \circ \delta_i$, and $s_i : S_n \to S_{n+1}$ by $f \mapsto f \circ \sigma_i$. We can check that the commutation relations are satisfied, and thus the sets $\{S_n\}$ form a simplicial set, and we have in fact defined a functor from the category of topological spaces into the category of simplicial sets.

Now we have a sequence of functors, all of which can be shown to be homotopy-invariant:

$$\text{Spaces} \xrightarrow{S_*} \text{Simplicial Sets} \longrightarrow \text{Simplicial Abelian Groups} \longrightarrow \text{Chain Complexes} .$$

This sequence shows homotopic maps induce induce identical maps on homology.

2. Homology

To better understand homology, we consider the following sequnece of abelian groups:

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 .$$

If we require $g \circ f = 0$, i.e., $\text{Im}(f) \subset \ker(g)$, then we have a chain complex and homology groups are defined.

We say the sequence is *exact* (at B) if we have $\ker(g) = \text{Im}(f)$. This condition is to equivalent to saying the homology group at B is trivial, and thus homology groups of a chain complex measures the "non-exactness" of the complex.

Exercise 2.1. Check that the condition that the above complex is exact at A is equivalent to f being injective, and the complex being exact at C is equivalent to g being surjective.

If a sequence of abelian groups is exact at every group, then we say it is an *exact sequence*. One particularly useful sequence is the one of the form presented above, and if it is exact then it is known as a *short exact sequence*.

Recall that the chain complexes form a category, where a map $f : C_* \to D_*$ between two complexes is defined to be a sequence of maps $\{f_i : C_i \to D_i\}$, such that the following diagrams commute:

$$\begin{array}{ccc} C_i & \xrightarrow{d} & C_{i-1} \\ \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle f_{i-1}} \\ D_i & \xrightarrow{d} & D_{i-1}. \end{array}$$

We say the sequence of chain complexes

$$0 \longrightarrow C'_* \xrightarrow{f} C_* \xrightarrow{g} C''_* \longrightarrow 0$$

is exact if in the following (big) commutative diagram, each column is a short exact sequence (of abelian groups):

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & C'_{n+1} & \longrightarrow & C'_n & \longrightarrow & C'_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\
\cdots & \longrightarrow & C_{n+1} & \longrightarrow & C_n & \longrightarrow & C_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle g_{n+1}} & & \downarrow{\scriptstyle g_n} & & \downarrow{\scriptstyle g_{n-1}} & & \\
\cdots & \longrightarrow & C''_{n+1} & \longrightarrow & C''_n & \longrightarrow & C''_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The most fundamental result of homological algebra tells us there is a nice relation between the homology groups in this case.

Theorem 2.2. *Suppose we have a short exact sequence of chain complexes,*

$$0 \longrightarrow C'_* \xrightarrow{f} C_* \xrightarrow{g} C''_* \longrightarrow 0 \; .$$

Then the following sequence is exact:

$$\cdots \longrightarrow H_{n+1}(C'') \xrightarrow{\delta} H_n(C') \xrightarrow{f_*} H_n(C) \xrightarrow{g_*} H_n(C'') \xrightarrow{\delta} H_{n-1}(C') \longrightarrow \cdots ,$$

where f_*, g_* are the induced maps, and δ is a mysterious (yet well-defined) map known as the connecting homomorphism.

This sequence of homology groups is a *long exact sequence.* As we will see later, this long exact sequence will enable us to compute the homology groups of various familiar topological spaces, for example the $n-$sphere.