FINITE CATEGORIES

NOTES FOR THE REU: JULY 29
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For clarity, we state first the definition of homotopy equivalence:

Definition 0.1. Given two spaces, X and Y, and two maps $f : X \to Y$ and $g : Y \to X$, we say that f and g are *inverse homotopy equivalences* if both composites are homotopic to the identity maps. That is,

$$g \circ f \simeq id_X \text{ and } f \circ g \simeq id_Y$$

Previously we defined a series of functors

$$\mathcal{S}paces \to \mathcal{S}implicial\ \mathcal{S}ets \to \mathcal{S}implicial\ \mathcal{A}belian\ \mathcal{G}roups \to \mathcal{C}hain\ \mathcal{C}omplexes$$

taking

$$(X \xrightarrow{f} Y) \mapsto (S_*X \xrightarrow{f} S_*Y) \mapsto (A_*X \xrightarrow{f} A_*Y) \mapsto (C_*X \xrightarrow{f} C_*Y)$$

and we noted that the product $X \times I$ gives

$$X \times I \mapsto S_*(X \times I) \mapsto A_*X \otimes I \mapsto C_*X \otimes I$$

A chain homotopy is defined precisely so that a homotopy of topological spaces produces, under these functors, a chain homotopy of complexes. Then the remainder of the result is given by a theorem.

Theorem 0.2. *If $f_* : C_* \to D_*$ and $g : D_* \to C_*$ are chain homotopic maps, then f_* and g_* induce inverse isomorphisms on the homology groups $H_*(C)$ and $H_*(D)$.*

Example 0.3. Suppose that the unique map $X \to pt.$ and an inclusion $pt \to X$ are inverse homotopy equivalences, where pt is the one-point space (i.e. X is *contractible*). Then $H_n(X) = H_n(pt) = 0$ for $n \neq 0$, and $H_0(X) = H_0(pt) = \mathbb{Z}$.

Exercise 0.4. Show that $D^n = \{x \in \mathbb{R}^n \big| |x| \leq 1\}$ is contractible.
Hint: consider the map $h : D^n \times I \to D^n$ given by $h(x,t) = tx$.

The following proposition is one of the foundational tools for computing homology.

Proposition 0.5. *A short exact sequence of chain complexes*

$$0 \to C'_* \xrightarrow{f_*} C_* \xrightarrow{g_*} C''_* \to 0$$

gives rise to a long exact sequence of homology groups

$$\cdots \to H_n(C') \xrightarrow{f_*} H_n(C) \xrightarrow{g_*} H_n(C'') \xrightarrow{\partial} H_{n-1}(C') \to \cdots .$$

Proof. The only difficult part of this theorem is paying attention long enough to check all of the details. The major first step is the definition of ∂, which we give now.

Suppose $x'' \in Z''_n$. By the surjectivity of g_n, $\exists\, x \in C_n$ such that $g_n(x) = x''$. Now since g_* commutes with the differentials, $g_{n-1}d_n(x) = d''_n g_n(x) = d''_n(x'') = 0$. Therefore, by exactness at C_*, $\exists\, x' \in C'_{n-1}$ such that $f_{n-1}(x') = d_n(x)$. Now since f_* commutes with the differentials, $f_{n-1}d'_{n-1}(x') = d_n f_{n-1}(x') = d_n^2(x) = 0$. So $d'_{n-1}(x') \in Z'_{n-1}$. We define $\partial(x'')$ to be the homology class of $d'_{n-1}(x')$.

What remains now is to check that ∂ is a well-defined homomorphism of apelian groups, and that the sequence above is exact at all stages. None of these are particularly difficult, they do require patience and care and are (as always) left to the reader.

As an example of the usefulness of a long exact sequence, we note that one can immediately compute the homology of all spheres inductively. For this we introduce the *reduced* homology groups. For any space X, the reduced homology $\widetilde{H}_n(X)$ is defined so that $H_n(X) = \widetilde{H}_n(X) \oplus \mathbb{Z}$, where $\mathbb{Z}$ is in degree zero. Then there is a similar long exact sequence for quotients; if $A \subset X$ is a sub-simplicial complex, then we have a long exact sequence

$$\cdots \to \widetilde{H}_n(A) \to \widetilde{H}_n(X) \to \widetilde{H}_n(X/A) \to \widetilde{H}_{n-1}(A) \to \cdots$$

Example 0.6. The n-sphere S^n is a quotient D^n/S^{n-1}, and applying the sequence above we have

$$\cdots \to \widetilde{H}_q(S^{n-1}) \to \widetilde{H}_q(D^n) \to \widetilde{H}_q(S^n) \to \widetilde{H}_{q-1}(S^{n-1}) \to \widetilde{H}_q - 1(D^n) \cdots$$

now recalling that $\widetilde{H}_q(D^n) = 0\ \forall\ n$, we have that

$$\widetilde{H}_q(S^n) \cong \widetilde{H}_{q-1}(S^{n-1}) \text{ for all } q$$

Thus, inductively, $\widetilde{H}_n(S^n) \cong \mathbb{Z}$ and $\widetilde{H}_q(S^n) = 0$ for $q \neq 0$.

Remark 0.7. This same method can be applied to any space X and its *suspension*, ΣX, to find

$$\widetilde{H}_q(\Sigma X) \cong \widetilde{H}_{q-1}(X) \text{ for all } q.$$