What Does Mathematical Logic Have To Do With Mathematics?

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July 13, 2005
The University of Chicago REU

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Abstract

Mathematical Logic is often seen as a peripheral subject in mathematics which deals only in pathology and foundational issues irrelevant to the real concerns of mathematicians. In this talk, I hope to convince you of the error in this point of view. Since we will be limited in time, we will confine our attention to the role of set theory and computability theory in the study of real numbers. We will see how these subjects provide new tools to solve problems in analysis and how they allow us to define interesting mathematically rich sets of real numbers. We will end by discussing how some natural mathematical questions in analysis turn out to be neither provable nor refutable from the normal axioms of mathematics, and how ridiculously large infinite sets may be able to come to the rescue.

1 Set Theory: Cardinals, Ordinals, and Computability Theory

What are open sets, closed sets?

Theorem 1.1. *A set $\mathcal{O} \subseteq \mathbb{R}$ is* ***open*** *iff $\mathcal{O}$ is a countable union of disjoint open intervals.*

Definition 1.1. *Let C be a closed set. (Cantor-Bedixson derivative of C)*

$$C' = C - \{\textit{isolated points of } C\}$$

Definition 1.2. *P is* ***perfect*** *if P is closed and has no isolated points.*

Idea: Take a closed C. Look at $C, C', C'', \ldots, C^n$.

Definition 1.3. $C^\alpha = \bigcap_n C^n$

Theorem 1.2. *If C is closed, there is some ordinal α such that C^α is perfect.*

Corollary 1.1. *If C is closed, then there is a perfect set P and a countable set A such that $C = A \cup P$ with $A \cap P = \emptyset$.*

Infinite sequences of 0's and 1's

0110111100101...

- No observable pattern
- Unpredictable
- It's hard to compress intitial segment 0101010101...

1.1 Computability Theory

There is a precise definition of "$f : \mathbb{N} \to \mathbb{N}$ is computable." (Use of Turing Machines)

Definition 1.4. *A function f from a finite sequence of 0's and 1's to $\mathbb{R}^+ \cup \{0\}$ is a* ***betting strategy*** *if*

$$f(\sigma) = \frac{f(\sigma 0) + f(\sigma 1)}{2}$$

Definition 1.5. *A betting strategy f is* ***effective*** *if we can computably approximate $f(\sigma)$ given σ.*

Definition 1.6. *A strategy for the first player is a function from sequences of even length to {0,1}. A strategy for the second player is a function from sequences of odd length to {0,1}.*

Definition 1.7. *A strategy is* ***winning*** *if whenever that player follows it, they win the game.*

Definition 1.8. *G is* ***determined*** *if some player has a winning strategy.*

Question: Is every game determined?

Answer: No. Using the Axiom of Choice, one can do a big diagnolization against all possible strategies to construct a very pathological game.

General Feeling: Every "nice" game is determined.

Theorem 1.3. *G Open $\Rightarrow G$ is determined.*

Proof. Suppose player one does not have a winning strategy. We define a strategy for player two, and avoid certain defeat. □

Definition 1.9. *The collection of* ***Borel*** *sets is the smallest collection of sets containing all open sets and closed under complementation and countable unions.*

Theorem 1.4. *Every Borel set is determined.*

Theorem 1.5. *You need really really big sets to prove this.*

Fact: The statement "Every projective set is determined" is independent of the usual axioms of set theory. It turns out that if you add axioms to set theory asserting the existence of rediculously large sets (large cardinals), then every projective set is determined (measurable).