ASYMPTOTIC BEHAVIOR OF THE NUMBER OF PRIME PARTITIONS

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1. Introduction

A prime partition of a positive integer n is a collection of positive integers $\{a_1, a_2, ..., a_k\}$, not necessarily distinct, so that every a_i is either 1 or a prime, and $a_1 + a_2 + ... + a_k = n$. The number of different prime partitions (with the same terms in different orders counted as equivalent) is denoted $q(n)$.

Theorem 1.

$$\ln q(n) = \theta\left(\sqrt{\frac{n}{\ln n}}\right)$$

2. A Lower Bound

Proposition 2.

$$\ln q(n) \geq 2\sqrt{2}\sqrt{\frac{n}{\ln n}}$$

Proof. One specific kind of prime partition of n is the kind in which all primes are distinct and less than $x(n)$, where $x(n)$ is the largest prime such that $2 + 3 + 7 + \cdots + x \leq n$. Call this particular prime partition a "simple prime partition." The number of simple prime partitions is less than $q(n)$, so an approximation would yield a lower bound for $q(n)$. There are exactly $2^{\pi(x(n))}$ such prime partitions, where $\pi(x(n))$ is the number of primes less than or equal to $x(n)$. This is because every prime partition whose primes are all distinct and less than $x(n)$ is formed by taking a subset of the primes from 2 to $x(n)$, and adding enough ones to make the total sum equal to n. Now $\ln(2^{\pi(x(n))}) = \theta(\pi(x(n))$ so it is enough to show $\pi(x(n)) = \theta(\sqrt{\frac{n}{\ln n}})$.

Lemma 3.

$$\sum_{p\leq x} p \sim \frac{x^2}{2\ln x}$$

Proof. By a consequence of the Prime Number Theorem, the nth prime $p_n \sim n \ln n$.

$$\sum_{p \le x} p \sim \sum_{n=1}^{\pi(x)} n \ln n = 1 \ln 1 + 2 \ln 2 + \dots \pi(x) \ln \pi(x).$$

We also know by the Prime Number Theorem that $\pi(x) \sim \frac{x}{\ln x}$. So

$$\sum_{n=1}^{\pi(x)} n \ln n \sim \sum_{n=1}^{\frac{x}{lnx}} n \ln n.$$

For large x, this approaches

$$\int_1^{\frac{x}{\ln x}} n \ln n \, dn = \frac{\frac{x^2}{(\ln x)^2} \ln(\frac{x}{\ln x})}{2} - \frac{x^2}{4(\ln x)^2} + \frac{1}{4} \sim \frac{x^2}{2 \ln x} - \frac{x^2}{4(\ln x)^2} \sim \frac{x^2}{2 \ln x}.$$

□

Now use this identity to show that $\pi(x(n)) \sim 2\sqrt{2}\sqrt{\frac{n}{\ln n}}$. It is true that $x(n) \sim \sqrt{2n \ln n}$ since composing this function with $\frac{x^2}{2 \ln x}$ we get

$$\frac{\sqrt{2n \ln n}^2}{2 \ln \sqrt{2n \ln n}} = \frac{2n \ln n}{2 \ln(n \ln n))} \sim n.$$

Substituting, we get

$$\pi(x(n)) \sim \pi(\sqrt{2n \ln n})$$

$$\pi(x(n)) \sim \frac{\sqrt{2n \ln n}}{\ln \sqrt{2n \ln n}} = 2 \, \frac{\sqrt{2n \ln n}}{\ln 2 + \ln n + \ln \ln n}$$

$$\sim 2 \frac{\sqrt{2n \ln n}}{\ln n} \sim 2\sqrt{2}\sqrt{\frac{n}{\ln n}}$$

□

3. An Upper Bound

Proposition 4.

$$q(n) \le \sqrt{2} \ln\left(\frac{5e}{\sqrt{2}}\right)\sqrt{\frac{n}{\ln n}}$$

Proof. Here is one way to construct an upper bound for $q(n)$: Choose n primes, with repetition permitted, of the primes between 0 and $x(n)$. Add the number of such choices to the number of choices of $\frac{n}{x(n)}$ primes between 0 and $2x(n)$, then choices of $\frac{n}{2x(n)}$ primes between 0 and $4x(n)$, and continue, multiplying these quantities until we reach a power of two for which $2^k x(n) > n$. The number of primes chosen within each block

$[2^k x(n), 2^{k+1}x(n)]$ is $\frac{n}{2^k x(n)}$ because if every prime is greater than 2^k, then any sum of more than $\frac{n}{2^k x(n)}$ will automatically be too large to be a partition of n.

What is the highest allowable power of two? If

$$2^k\sqrt{2n\ln n} = n,$$

then

$$k = \frac{\ln(\frac{n}{\sqrt{2n\ln n}})}{\ln 2} \sim \frac{\ln n}{2\ln 2}.$$

So, using the fact that the number of ways to choose k objects at a time from n is $\binom{n+k-1}{k}$, we can write a product expression for this upper bound.

$$q(n) \le \prod_{k=0}^{\frac{\ln n}{2\ln 2}} \binom{\pi(2^{k+1}x(n)) - \pi(2^k x(n)) + \frac{n}{2^k x(n)} - 1}{\frac{n}{2^k x(n)}}$$

Use the fact (which will be proven in 5) that

$$\binom{n}{k} < \left(\frac{en}{k}\right)^k.$$

If we let $z = 2^k x(n)$, this yields

$$q(n) \le \prod_{k=0}^{\frac{\ln n}{2\ln 2}} e^{\frac{n}{z}}\left(\frac{z^2}{n\ln z} + 1 - \frac{z}{n}\right)^{\frac{n}{z}}$$

Note that $\frac{n}{z} = \frac{1}{\sqrt{2}}\sqrt{\frac{n}{\ln n}}$ and that $\frac{z}{\ln z} \sim 2^{k+2}\frac{n}{z}$. So the kth term of the product is asymptotically equal to

$$\left(\frac{5e(\sqrt{\frac{n}{2\ln n}})}{\sqrt{\frac{n}{\ln n}}}\right)^{\frac{1}{2^k}\sqrt{\frac{n}{2\ln n}}}$$

Now let y be $\sqrt{\frac{n}{2\ln n}}$. This makes the kth term of the product equal to

$$= \left(\frac{5e(\sqrt{\frac{n}{2\ln n}})}{\sqrt{\frac{n}{\ln n}}}\right)^{\frac{1}{2^k}\sqrt{\frac{n}{2\ln n}}}$$

$$= \left(\frac{5ey}{\sqrt{2}y}\right)^{\frac{1}{2^k}y}$$

$$= \left(\frac{5e}{2}\right)^{\frac{1}{2^k}y}.$$

Now when we take the natural log of this, we should have a constant multiple of $\sqrt{\frac{n}{\ln n}}$. We see that

$$\ln \prod_{k=0}^{\frac{\ln n}{2\ln 2}} (\frac{5e}{\sqrt{2}})^{\frac{1}{2^k}\sqrt{\frac{n}{2\ln n}}} = \sum_{k=0}^{\frac{\ln n}{2\ln 2}} \frac{1}{2^k}\sqrt{\frac{n}{2\ln n}}\ln\frac{5e}{\sqrt{2}}$$

$$\sim \sqrt{2}\ln\left(\frac{5e}{\sqrt{2}}\right)\sqrt{\frac{n}{\ln n}}$$

□

4. Proof of an Upper Bound of the Binomial Coefficient

Proposition 5.

$$\binom{n}{k} < (\frac{ae}{b})^b$$

Proof. The Taylor expansion of e^x is

$$e^x = 1 + x + \frac{x^2}{2!_{\frac{x^3}{3!}}} + \cdots + \frac{x^k}{k!} + \dots$$

Therefore, for any individual k,

$$\frac{x^k}{k!} < e^x.$$

In particular, letting $x = k$,

$$\frac{k^k}{k!} < e^k$$

or

$$1 < \frac{ke^k}{k^k}$$

Now observe that

$$\frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)...(n-k+1)}{k!} \leq \frac{n^k}{k!}.$$

Recalling that the previous expression was greater than one, we have

$$\binom{n}{k} < \frac{n^k}{k!}\frac{k!e^k}{k^k} = \left(\frac{ne}{k}\right)^k.$$

□

5. An Upper Bound for q(n) using Generating Functions

Another way to estimate partitions is using generating functions. The Hardy-Ramanujan formula for p(n), the number of unrestricted prime partitions, was based on the fact that p(n) is the coefficient of x^n in the product

$$\Pi \frac{1}{1-x^n}$$

where x is between 0 and 1. There is a similar generating function for $q(n)$, that is,

$$\Pi \frac{1}{1-x^p}$$

where p is a prime less than n and x is between 0 and 1. Expanding this sum it becomes clear that for every prime partition there is a corresponding contribution to the x^n term. Now note that

$$q(n) \leq \frac{1}{x^n} \Pi \frac{1}{1-x^p}$$

since there are other terms in the product than the x^n term. So

$$\ln q(n) \leq -\ln x + \sum_{p \leq n} \frac{1}{1-x^p}.$$

And we claim this is $\theta(\sqrt{\frac{x}{\ln x}})$, so at least the first term must be of that order. This gives us a generous necessary value for x;

$$-n \ln x = \theta(\sqrt{\frac{n}{\ln n}})$$

$$x = e^{-\frac{1}{\sqrt{n \ln n}}} \sim 1 - \frac{1}{\sqrt{n \ln n}}.$$

Now it's left to calculate the sum, and in future we will keep

$$x = 1 - \frac{1}{\sqrt{n \ln n}}.$$

Now consider

$$\sum_{p \leq n} \ln \frac{1}{1-x^p}.$$

We will prove that this is $\theta(\sqrt{\frac{n}{\ln n}})$, but we will not keep track of the actual constant factor, and the symbol $\sim$ will be used to mean "asymptotically equivalent, possibly with a constant factor." Note that

$$\ln \frac{1}{x} \sim \frac{1}{\sqrt{n \ln n}}.$$

Also note that

$$\frac{1}{1-x} = \sqrt{n\ln n}$$

so

$$\ln\frac{1}{1-x} \sim \ln n.$$

We divide the sum into three cases:

$$\textbf{Case 1}: p < \sqrt{\frac{n}{\ln n}}$$

$$\textbf{Case 2}: \sqrt{\frac{n}{\ln n}} \leq p \leq \sqrt{n\ln n}$$

$$\textbf{Case 3}: p > \sqrt{n\ln n}$$

For simplicity, let

$$M_1 = \sqrt{\frac{n}{\ln n}}$$

and

$$M_2 = \sqrt{n\ln n}.$$

Lemma 6 (Case 1)**.**

$$\sum_{p\leq M_1} \ln\frac{1}{1-x^p} \sim \sqrt{\frac{n}{\ln n}}.$$

Proof. In this case,

$$p \leq \sqrt{\frac{n}{\ln n}},$$

so since $\frac{1}{1-x^p}$ is a decreasing function of p,

$$\sum_{p\leq M_1} \ln\frac{1}{1-x^p} \leq \ln\frac{1}{1-x^2}\pi(\sqrt{\frac{n}{\ln n}}).$$

Using the Prime number Theorem, this yields

$$(\ln\frac{1}{1-x} + \ln\frac{1}{1+x})\frac{\sqrt{\frac{n}{\ln n}}}{\ln\sqrt{\frac{n}{\ln n}}}.$$

As n becomes large, x approaches 1, so $\ln\frac{1}{1+x}$ approaches 0. Thus the previous expression is equivalent to

$$(\ln n + 0)\frac{\sqrt{\frac{n}{\ln n}}}{\ln\sqrt{\frac{n}{\ln n}}} \sim \ln x\frac{\sqrt{n}}{(\ln n)^{3/2}}$$

$$= \sqrt{\frac{n}{\ln n}}.$$

□

Lemma 7 (Case 2)**.**

$$\sum_{p=M_1}^{M_2} \ln \frac{1}{1-x^p} \sim \sqrt{\frac{n}{\ln n}}.$$

Proof. We split this expression into a dyadic sum, breaking it up into powers of e and replacing it with

$$\sum_{k=\ln M_1}^{\ln M_2} \sum_{p=e^k}^{e^{k+1}} \ln \frac{1}{1-x^{e^p}}.$$

The expression inside the nested sum is a decreasing function of p, so the sum is less than its highest value multiplied by the number of terms:

$$\sum_{k=\ln M_1}^{\ln M_2} \ln \frac{1}{1-x^{e^k}} \pi(e^{k+1}).$$

Using the prime number theorem, this is asymptotically equivalent to

$$\sum_{k=\ln M_1}^{\ln M_2} \ln \frac{1}{1-x^{e^k}} \frac{e^{k+1}}{k+1}$$

Since k is at least $\ln M_1$, this sum is less than or equal to

$$\frac{1}{\ln M_1} \sum_{k=\ln M_1}^{\ln M_2} \ln \frac{1}{1-x^{e^k}} e^{k+1}$$

$$\leq \frac{1}{\ln M_1} \sum_{k=\ln M_1}^{\ln M_2} \int_{k-1}^{k} \ln(\frac{1}{1-e^{x^t}}) e^{t+2} dt$$

$$= \frac{e^2}{\ln M_1} \int_{\ln M_1}^{\ln M_2} \ln \frac{e^t}{1-x^{e^t}} dt.$$

Computing this integral, we find it equal to a constant times

$$\frac{1}{\ln M_1} \int_{M_1}^{M_2} \ln \frac{1}{1-x^u} du$$

$$= \frac{1}{\ln M_1} \frac{1}{\ln x} \int_{x_2^M}^{x_1^M} \ln \frac{1}{1-v} \frac{dv}{v}$$

$$\sim \frac{1}{\ln n} * \sqrt{n \ln n} * \int_{x^{M_2}}^{x^{M_1}} \ln \frac{1}{1-v} \frac{dv}{v}$$

But the integral is less than a constant. Because $\ln(1-v)$ is close to zero for these values of v, and $\frac{1}{v}$ is bounded since $x^{M_2} \geq e^{-1}$.so the entire expression is less than or equal to

$$\frac{1}{\ln n} * \sqrt{n \ln n} = \sqrt{\frac{n}{\ln n}}.$$

□

Lemma 8 (Case 3)**.**

$$\sum_{p=M_2}^{n} \ln \frac{1}{1-x^p} \sim \sqrt{\frac{n}{\ln n}}.$$

Proof. Note that

$$\ln \frac{1}{1-x^p} \leq \frac{x^p}{1-x^p}$$

since $\ln \frac{1}{1-x^p} = \ln(1 + \frac{x^p}{1-x^p}) \leq \frac{x^p}{1-x^p}$. Again we construct a dyadic sum.

$$\sum_{p=M_2}^{n} \frac{x^p}{1-x^p} = \sum_{k=1+\ln M_2}^{\ln n} \sum_{p=k-1}^{k} \frac{x^{e^p}}{1-x^{e^p}}$$

This is a decreasing function of p, so the inner sum is less than its largest term times the number of terms:

$$\leq \sum_{k=1+\ln M_2}^{\ln n} \frac{x^{e^k}}{1-x^{e^k}} \pi(e^k)$$

Using the Prime Number Theorem, this is asymptotically equal to

$$\sum_{k=1+\ln M_2}^{\ln n} \frac{x^{e^k}}{1-x^{e^k}} \frac{e^k}{k}.$$

Since $k \geq 1 + \ln M_2$, the previous expression is less than if we divided by the largest possible value of k, so

$$\leq \frac{1}{1+\ln M_2} \sum_{k=1+\ln M_2}^{\ln n} \frac{x^{e^k}}{1-x^{e^k}} e^k$$

Now, what's inside the sum is a monotonically increasing function, so it is less than the integral:

$$\leq \frac{1}{1+\ln M_2} \int_{2+\ln M_2}^{1+\ln n} \frac{x^{e^t}}{1-x^{e^t}} e^t dt$$

We calculate by substitution:

$$\begin{aligned}
&\frac{1}{1+\ln M_2}\int_{e^2 M_2}^{en}\frac{x^u}{1-x^u}du\\
&=\frac{1}{1+\ln M_2}*\frac{1}{\ln x}\int_{x^{e^2 M_2}}^{x^{en}}\frac{dy}{1-y}\\
&\sim\frac{1}{1+\ln M_2}*\frac{1}{\ln x}ln(\frac{1-x^{e^2 M_2}}{1-x^{en}})\\
&\leq\frac{1}{1+\ln M_2}*\frac{1}{\ln x}\ln(1-x^{e^2 M_2})\\
&\sim\sqrt{n\ln n}*\frac{1}{\ln n}\ln(1-x^{e^2 M_2})\\
&\sim\sqrt{\frac{n}{\ln n}}\ln(1-x^{e^2 M_2}).
\end{aligned}$$

Now $\ln(1-x^{e^2 M_2})$ is bounded by a constant, since

$$x^p=(1-\frac{1}{\sqrt{n\ln n}})^{\sqrt{n\ln n}\frac{p}{\sqrt{n\ln n}}}\sim(e^{-1})^{\frac{p}{\sqrt{n\ln n}}}$$

Therefore, the entire sum is less than or asymptotically equal to

$$\sqrt{\frac{n}{\ln n}}*C$$

so it's of the order of

$$\sqrt{\frac{n}{\ln n}}.$$

□

Since all three sections of the sum are of the order of $\sqrt{\frac{n}{\ln n}}$, the entire sum is of that order, and the upper bound is justified.

References

[1] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers.* Clarendon Press, Oxford, 1979.

[2] I am very grateful to Prof. Laszlo Babai, for suggesting this problem and guiding my work; to Emma Smith, my graduate mentor, for her advice; and to Prof. Peter Constantin, for teaching me the technique of dyadic sums.