Subgroups of Mapping Class Groups Generated by Three Dehn Twists

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Abstract

It has been shown that the mapping class group of a genus $g \geq 2$ closed surface can be generated by $2g+1$ Dehn twists. Little is known, however, about the structure of those mapping class groups. This paper is an attempt to explain the structure of subgroups of the mapping class group of a closed surface of genus $g \geq 2$ which are generated by three Dehn twists.

1 Background and Notation

This paper stems from work done for Benson Farb's Topology class during the summer 2006 University of Chicago REU, and most of the necessary background is contained in the first three chapters of [2]. Also, some familiarity with the braid group on n strands is necessary, and an excellent introduction is [1].

For notational consistency and convenience, let Mod_g denote the mapping class group of a genus g surface, T_a denote the Dehn twist about the simple closed curve a, $i(a, b)$ denote the geometric intersection number of the simple closed curves a and b, and (x, y, z) denote the intersection numbers of three Dehn twists as follows: $x = i(a, b)$, $y = i(a, c)$, $z = i(b, c)$. Finally, let $\sigma_1, \sigma_2, ..., \sigma_{n-1}$ be the standard generators of B_n, the braid group on n strands.

Three key concepts used throughout this paper are the Change of Coordinates Principle, some Dehn twist basics, and subgroups generated by two Dehn twists.

The change of coordinates principle states that to find, for example, the subgroup generated by the Dehn twists T_a, T_b, and T_c, where $i(a,b) = x$, $i(a,c) = y$ and $i(b,c) = z$, we need only consider one such triple of Dehn twists T_a, T_b, and T_c, such that $i(a,b) = x$, $i(a,c) = y$ and $i(b,c) = z$. Hence the notation for a trio of Dehn twists noted below contains all the necessary information about a designated triple.

The most crucial Dehn twist basics we shall use are:

- Fact 1 $i(a,b) = 0 \Rightarrow T_a T_b = T_b T_a$
- Fact 2 $i(a,b) = 1 \Rightarrow T_a T_b T_a = T_b T_a T_b$
- Fact 3 given any $f \in Mod_g$, $T_{f(a)} = f T_a f^{-1}$
- Fact 4 $i(T_a^k(b), b) = |k| i(a,b)^2$ where $k \in Z$.

More information and proofs of both the change of coordinates principle and the Dehn twist basics are covered in [2, pp. 33-35] and [2, pp. 50-52] respectively.

A final crucial bit of background is the groups generated by two Dehn twists. In [3] Ishida proves

Theorem 1.1 *given a genus $g \geq 2$ surface,*

- *if $i(a,b) = 0$ then $< T_a, T_b > \cong Z^2$,*
- *if $i(a,b) = 1$ then $< T_a, T_b > \cong B_3$,*
- *if $i(a,b) \geq 2$ then $< T_a, T_b > \cong F_2$,*

where F_2 is the free group on two generators.

The proofs for the first two are simple and follow from Fact 1 and Fact 2, whereas the proof of the third statement is a good deal longer and more difficult, so the reader is directed to [3]. From the Theorem 1.1 it follows that:

Corollary 1.2 $< \sigma_1, \sigma_2^2 \sigma_1 \sigma_2^{-2} > \cong F_2$

Proof Given T_a and T_b such that $i(a,b) = 1$, by Theorem 1.1, $\exists \varphi \colon < T_a, T_b > \to B_3$, an isomorphism. Without loss of generality, let $\varphi \colon T_a \to \sigma_1$ and $\varphi \colon T_b \to \sigma_2$. It is clear by Fact 4 that $i(a, T_b^2(a)) = 2$, thus by Theorem 1.1 $< T_a, T_{T_b^2(a)} > \cong F_2$. By Fact 3 $T_{T_b^2(a)} = T_b^2 T_a T_b^{-2}$, hence $\varphi \colon T_{T_b^2(a)} \to \sigma_2^2 \sigma_1 \sigma_2^{-2}$. Therefore $F_2 \cong < T_a, T_{T_b^2(a)} > \cong < T_a, T_b^2 T_a T_b^{-2} > \cong < \sigma_1, \sigma_2^2 \sigma_1 \sigma_2^{-2} >$. ∎

2 Three Subgroups Generated by Three Dehn Twists

After stating the results of the theorem by Ishida, Farb and Margalit note [2, p. 53] that the corresponding question for three Dehn twists is totally open. This section addresses three cases, $(1,0,0)$, $(2,0,0)$, and $(1,1,1)$.

Theorem 2.1 *Given* T_a, T_b, *and* T_c *such that* $(1,0,0)$, $< T_a, T_b, T_c > \cong B_3 \times Z$

Proof Since $i(a,c) = i(b,c) = 0$, $T_aT_c = T_cT_a$ and $T_bT_c = T_cT_b$ by Fact 1. Hence $\forall f \in < T_a, T_b, T_c >$, $f = gT_c^n$ where $g \in < T_a, T_b >$ and $n \in Z$. Hence $< T_a, T_b, T_c > \cong < T_a, T_b > \times Z$ via the map ψ where $\psi\colon f \to (g,n)$. By Theorem 1.1 $i(a,b) = 1 \Rightarrow < T_a, T_b > \cong B_3$ thus $< T_a, T_b > \times Z \cong B_3 \times Z$. ∎

In a similar vein to $(1,0,0)$ is the subgroup generated by a trio of Dehn twists such that $(2,0,0)$.

Theorem 2.2 *Given* T_a, T_b, *and* T_c *such that* $(2,0,0)$, $< T_a, T_b, T_c > \cong F_2 \times Z$

Proof Since $i(a,c) = i(b,c) = 0$, $T_aT_c = T_cT_a$ and $T_bT_c = T_cT_b$ by Fact 1. Hence $\forall f \in < T_a, T_b, T_c >$, $f = gT_c^n$ where $g \in < T_a, T_b >$ and $n \in Z$. Hence $< T_a, T_b, T_c > \cong < T_a, T_b > \times Z$ via the map ψ where $\psi\colon f \to (g,n)$. By Theorem 1.1 $i(a,b) = 2 \Rightarrow < T_a, T_b > \cong F_2$ thus $< T_a, T_b > \times Z \cong F_2 \times Z$. ∎

Theorem 2.3 *Given* T_a, T_b, *and* T_c *such that* $(1,1,1)$, $< T_a, T_b, T_c > \cong B_3$

Proof By Theorem 1.1 $i(a,b) = 1 \Rightarrow < T_a, T_b > \cong B_3$ and by Fact 4 $i(a,b) = 1 \Rightarrow i(a, T_b(a)) = i(b, T_b(a)) = 1$. Fact 3 implies that $T_{T_b(a)} = T_bT_aT_b^{-1}$. Without loss of generality, if φ is the isomorphism $\varphi\colon < T_a, T_b > \to B_3$, then $\varphi(T_a) = \sigma_1$, $\varphi(T_b) = \sigma_2 \Rightarrow \varphi(T_{T_b(a)}) = \sigma_2\sigma_1\sigma_2^{-1}$. Since T_a, T_b, $T_{T_b(a)}$ are such that $(1,1,1)$, by the change of coordinates principle $T_{T_b(a)} = T_c$, hence $< T_a, T_b, T_c > \cong < \sigma_1, \sigma_2, \sigma_2\sigma_1\sigma_2^{-1} > \cong < \sigma_1, \sigma_2 > \cong B_3$. ∎

3 Other Subgroups Generated by Three Dehn Twists

This section addresses the cases $(1,1,1)$, $(2,1,1)$, $(2,2,0)$, $(2,1,0)$, $(2,2,1)$, and $(2,2,2)$ with two propositions regarding chains of simple closed curves.

Proposition 3.1 *The cases* $(2,1,1)$*, and* $(2,2,0)$ *are determined by the case of the chain* $(1,0,1)$*.*

Proof Let T_a, T_b, and T_c be such that $(1,0,1)$. Then $i(b,T_a^2(b)) = 2$ and $i(c,T_a^2(b)) = i(c,b) = 1$ by Fact 4. Thus $T_{T_a^2(b)}$, T_b, and T_c are such that $(2,1,1)$. By Fact 3, $T_{T_a^2(b)} = T_a^2 T_b T_a^{-2}$, thus the subgroup generated by the trio of Dehn twists with intersection $(2,1,1)$ is $< T_a^2 T_b T_a^{-2}, T_b, T_c >$. Considering the same T_a, T_b, and T_c be such that $(1,0,1)$, then by Fact 4 $i(a,T_b^2(c)) = i(c,T_b^2(c)) = 2$ and $i(a,c) = 0$. Thus $T_{T_b^2(c)}$, T_a, and T_c are such that $(2,2,0)$. By Fact 3, $T_{T_b^2(c)} = T_b^2 T_c T_b^{-2}$, thus the subgroup generated by the trio of Dehn twists with intersection $(2,2,0)$ is $< T_b^2 T_c T_b^{-2}, T_b, T_c >$. ∎

Proposition 3.2 *The cases* $(2,1,0)$*,* $(2,2,1)$*, and* $(2,2,2)$ *are determined by the case of the chain of four Dehn twists* T_a*,* T_b*,* T_c*, and* T_d *where* $i(a,b) = i(b,c) = i(c,d) = 1$*, and* $i(a,c) = i(a,d) = i(b,d) = 0$*.*

Proof Let T_a, T_b, T_c, and T_d as in the statement of the proposition. Then $i(b,T_c^2(d)) = 2$, $i(a,T_c^2(d)) = 0$, by Fact 4 and $i(a,b) = 1$. Thus T_b, $T_{T_c^2(d)}$, and T_a are such that $(2,1,0)$. By Fact 3 $T_{T_c^2(d)} = T_c^2 T_d T_c^{-2}$, thus the subgroup generated by the trio of Dehn twists with intersection $(2,1,0)$ is $< T_b, T_c^2 T_d T_c^{-2}, T_a >$. Considering the same T_a, T_b, T_c, and T_d, note that, by Fact 4, $i(b,T_c^2(d)) = i(T_a(b),T_c^2(d)) = 2$ and $i(b,T_a(b)) = 1$. Thus $T_{T_c^2(d)}$, T_b, and $T_{T_a(b)}$ are such that $(2,2,1)$. By Fact 3, $T_{T_c^2(d)} = T_c^2 T_d T_c^{-2}$ and $T_{T_a(b)} = T_a T_b T_a^{-1}$, thus the subgroup generated by the trio of Dehn twists with intersections $(2,2,1)$ is $< T_c^2 T_d T_c^{-2}, T_b, T_a T_b T_a^{-1} >$. Finally, note that $i(b,T_c^2(d)) = i(T_a^2(b),T_c^2(d)) = i(b,T_a^2(b)) = 2$ by Fact 4. Thus $T_{T_c^2(d)}$, T_b, and $T_{T_a^2(b)}$ are such that $(2,2,2)$. By Fact 3, $T_{T_c^2(d)} = T_c^2 T_d T_c^{-2}$ and $T_{T_a^2(b)} = T_a^2 T_b T_a^{-2}$, thus the subgroup generated by the trio of Dehn twists with intersections $(2,2,2)$ is $< T_c^2 T_d T_c^{-2}, T_b, T_a^2 T_b T_a^{-2} >$. ∎

4 Conjectures for Further Research

The question of the structure of subgroups of Mod_g generated by three Dehn twists with intersections $(x, y, z) \leq (2, 2, 2)$ has now been reduced to the problem of the subgroups generated by chains. These questions are still open, and even for Mod_2 for which Birman has calcuated a set of generators and relations [1, p. 184], the subgroups generated by these chains are non-trivial to derive, which leads to

Conjecture 4.1 *For a surface of genus $g \geq 2$, the subgroup generated by T_a, T_b, T_c, and T_d where $i(a, b) = i(b, c) = i(c, d) = 1$, and $i(a, c) = i(a, d) = i(b, d) = 0$ is B_5. Hence the subgroup generated by T_a, T_b, and T_c such that $(1, 0, 1)$ is B_4.*

It is clear that the relations for B_5 hold on $< T_a, T_b, T_c, T_d >$ but whether there are additional relations is open. In addition, this paper only explores intersections less than or equal to 2, hence

Conjecture 4.2 *Given T_a, T_b, and T_c such that (x, y, z), where x, y, $z \in \{0, 1, 2, 3, ...\}$ then $< T_a, T_b, T_c > \cong < T_l, T_m, T_n >$ for T_l, T_m, T_n such that (x_1, y_1, z_1) where x_1, y_1, $z_1 \in \{0, 1, 2\}$.*

As an example, the case $(x, 0, 0)$ where $x \geq 2$ is an immediate extension of Theorem 2.2 using the same proof.

References

[1] Joan S. Birman. *Braids, Links, and Mapping Class Groups.* Princeton University Press, Princeton, NJ, 1974.

[2] Benson Farb and Dan Margalit. A primer on mapping class groups. pre-print, 2006.

[3] Atsushi Ishida. The structure of subgroup of mapping class groups generated by two dehn twists. *Proc. Japan Acad. Ser. A*, 72:240–241, 1996.