ELEMENTARY MOVES ON POLYGONAL TRIANGULATIONS OF THE DISK

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1. Abstract

D. Sleator, R. Tarjan, and W. Thurston discovered that the diameter of the triangulation graph Γ_k was bounded above by $2k-10$.[2] I will show that for a pair of triangulations K and J to realize this distance, they must satisfy the following two conditions:

(1) K and J may share no diagonals
(2) No elementary move τ on K is such that τK shares a diagonal with J

I will also offer an elementary proof of a lemma from [2] stating that, if K and J share a diagonal, that diagonal is never moved in the shortest path from K to J in the triangulation graph.

2. Definitions

Definition 1 (2-Simplex). *A* 2-simplex *is a collection of three non-collinear points connected by line segments (i.e. a triangle). An n-simplex is defined analogously for dimensions other than 2, however I will only discuss the case when $n = 2$. The dimension of an n-simplex is n.*

Definition 2 (Simplicial Complex). *A* simplicial complex $\mathscr{S}$ *is a set of simplices such that for any two simplices $\gamma, \pi \in \mathscr{S}$, either $\gamma \cap \pi = \emptyset$ or $\gamma \cap \pi = \delta$ where $\delta \in \mathscr{S}$. The dimension of $\mathscr{S}$ will be the greatest dimension of simplex in $\mathscr{S}$.*

Definition 3 (Polygonal Triangulation). *Let K be a two-dimensional simplicial complex (i.e., a collection of triangles). If K is homeomorphic to a disk, then it is called a* triangulation *of a disk. Furthermore, if K is a k-gon divided by diagonals into $k-2$ triangles, then K will be called a* polygonal triangulation*. See Figure 1.*

A k-gon dissected into $k-2$ triangles will be called a k-triangulation for the purposes of this paper. Note that a k-triangulation has $k-3$ diagonals. The vertices of a triangulation K will be enumerated $v_1, \ldots, v_k$, moving clockwise around the outside of the k-gon.

Definition 4 (Outer Triangle). *If there is a vertex v_i such that there is an edge connecting v_{i-1} and v_{i+1}, then v_i will be called an* outer vertex*. The triangle $v_{i-1}v_iv_{i+1}$ will be called an* outer triangle*. The edge $v_{i-1}v_{i+1}$ is said to cut off the outer vertex v_i. It is clear that every polygonal triangulation of the disk has at least two outer triangles. In Figure 1, the vertices at the top and bottom are outer vertices.*

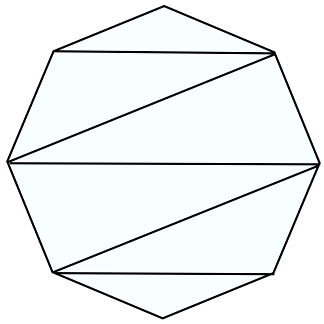

FIGURE 1. A polygonal 8-triangulation

Elementary moves (also called diagonal flips [4] or diagonal transformations [1]) are the desirable manipulations of triangulations for the problems under consideration since they preserve the structure of a triangulation. The definition contained in Negami's paper is as follows:

Definition 5 (Negami's Flip)**.** *Let abc and acd be two triangles in a triangulation K which share the common side ac. A* diagonal flip *is made by removing the segment ac and inserting the segment bd in the quadrilateral abcd. If the segment bd is already in K, then this diagonal flip cannot be made.*[4] *See Figure 2.*

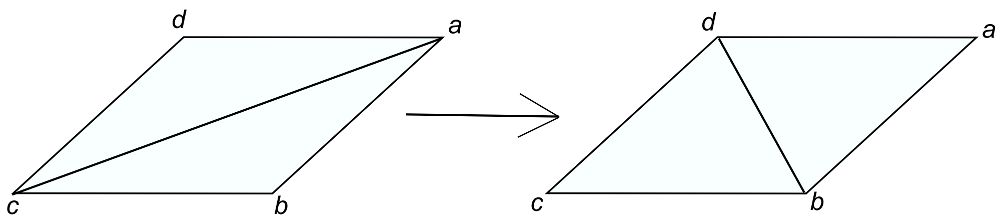

FIGURE 2. A diagonal flip

Nakamoto and Ota use the term diagonal transformation instead of diagonal flip. I will use the term *elementary move* interchangeably with diagonal transformation and diagonal flip.

If τ is an elementary move and K a triangulation, then the triangulation reached by performing τ on K will be K' if the specific elementary move used is clear or unimportant, or τK if the elementary move is to be stressed. Performing several elementary moves consecutively works the same way as composition of maps. If τ, α are elementary moves, then the composition, performing τ then α, will be written $(\alpha \circ \tau)K$, or simply $\alpha\tau K$. An elementary move may also be identified by the specific move of the diagonal, e.g. $\tau = ac \to bd$ is the elementary move on the quadrilateral $abcd$ that deletes ac and draws in bd.

Negami and others have previously described an equivalence relation on k-triangulations $K \sim J$ if J is the product of a series of diagonal transformations on K. That is, if $\exists\, \tau_1\,, \tau_2\,, \ldots\,, \tau_r$ elementary moves such that $J = \tau_r \circ \cdots \circ \tau_1 K$.

Definition 6 (Triangulation Graph)**.** *The* triangulation graph Γ_k *has as its vertices all k-triangulations of the disk. An edge is formed between two vertices K and J if $\exists\,\tau$ an elementary move such that $J = \tau K$.*

Definition 7 (Irreducible Cycles)**.** *A cycle $a = v_1, v_2, v_3, \ldots, v_{s-1}, v_s, v_1$ in the triangulation graph will be called irreducible if there does not exist a cycle $a' = v_1, v_2, w_1, \ldots, w_m, v_s, v_1$ such that a' is shorter than a.*

Definition 8 (Elementary Move Set)**.** *For the purposes of this paper, I will associate a set $EM(K)$ to the triangulation K, where $EM(K) = \{\tau_1, \ldots, \tau_r\}$ is the set of elementary moves that can be made on K. Two elementary moves τ and α will be called* disjoint *if $\tau \in EM(\alpha(K))$ and $\alpha \in EM(\tau(K))$. Otherwise, they will be said to be adjacent. It is clear that, for a k-triangulation K, $|EM(K)| = k - 3$.*

I will also sometimes discuss a triangulation as a graph itself. In this case, I will let $deg(v)$ denote the degree of a vertex v, and I will denote by $\dot{deg}_K(v)$ the interior degree of v in K. That is, the degree of v excluding the border of the polygon, or, equivalently, the number of diagonals with an end at v. The interior degree will be the more useful, since in polygonal triangulations, the border is fixed.

Definition 9 (Radial Triangulation)**.** *A polygonal triangulation K will be called a* radial triangulation *at v_i if all diagonal meet at a vertex, v_i. Equivalently, if $K \in \Gamma_k$, then a radial triangulation is a triangulation K in which $\exists\, v_i$ such that $\dot{deg}_K(v_i) = k - 3$. See Figure 3.*

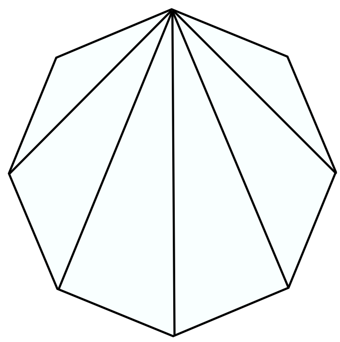

FIGURE 3. A radial triangulation

3. Formulation of the Problem

Recall that in graph theory, for a graph G, $V(G)$ generally denotes the set of vertices of G. Recall also that a genus g surface (denoted Σg) is the connect sum of g tori. Negami has proven the following theorem:

Theorem 10 (Negami's Theorem)**.** *For any genus g surface, $\exists\, N$ such that if K, K' are k-triangulations of Σ_g, $|V(K)| = |V(K')| > N$, then $K \sim K'$.* [4]

For this paper, the concept of a genus g surface is not required. Since I am only discussing triangulations of the disk, the following theorem suffices:

Theorem 11. *The graph Γ_k is connected for all k.*

D. Sleator, R. Tarjan and W. Thurston were the first to describe the triangulation graph [2]. They also improved on a previous upper bound for its diameter from $2k-6$ to $2k-10$, and proved that their new upper bound is also a lower bound. That is, there exist $K, J \in \Gamma_k$ such that the distance from K to J is $2k-10$.

An important open question is the following: given two triangulations K and J in Γ_k, what is the length of the shortest path connecting them? [2] begins to examine this question. They determined that, given triangulations K and J, if there is an elementary move $\tau \in EM(K)$ such that τK has a diagonal in common with J, then there is a minimal path from K to J in which τ is the first elementary move performed. But there are triangulations even as small as 6-triangulations in which K and J do not share any diagonals, nor would any elementary move on K set a diagonal that is contained in J in place. It is unclear in this situation how to construct the shortest path from K to J.

4. Properties of the Triangulation Graph

A few properties of the triangulation graph follow immediately from its construction:

Lemma 12. *Γ_k is regular of degree $k-3$ (i.e. every vertex in the triangulation graph is adjacent to exactly $k-3$ other vertices).*

Proof. This follows directly from the fact that $\forall K \in \Gamma_k$, $|EM(K)| = k-3$, since an edge in Γ_k corresponds with an elementary move on K. □

Lemma 13. *Γ_k contains k distinct (although not disjoint) subgraphs isomorphic to Γ_{k-1}.*

Proof. Such a subgraph can be found in Γ_k by choosing a triangulation, K. Then fix an outer triangle in K. More specifically, fix the diagonal. The remained of the k-gon is clearly a $(k-1)$-gon, and performing elementary moves (leaving our fixed diagonal in place) will create a subgraph of Γ_k that is isomorphic to Γ_{k-1}. As there are k vertices in a k-gon, this process may be performed k times (each time on a triangulation with a different outer vertex), yielding k subgraphs isomorphic to Γ_{k-1}. □

Lemma 14. *Every vertex of Γ_k is contained in at least two distint subgraphs of Γ_k which are isomorphic to Γ_{k-1}.*

Proof. It is enough that any k-triangulation has at least two outer vertices. Then each outer triangle can be fixed and the desired subgraph can be constructed as above. □

In [2], Negami's theorem was taken as proven. But Negami gives a complicated proof that is not necessary when only considering polygonal triangulations of the disk, as in [2] and here. The following lemma and theorem suffice:

Lemma 15. *Given a triangulation $K \in \Gamma_k$ and a vertex v of K, $\exists\, \tau_1, \ldots, \tau_r$ such that $\tau_r \ldots \tau_1 K = J$, where J is the radial triangulation at v and $r = k-3-\dot{deg}_K(v)$.*

Proof. This is meaningless for $k < 4$, and clearly true for $k = 4$. For $k > 4$, the claim follows from induction.

Assume the claim is true for $k-1$. Take K in Γ_k for $k > 4$. K has two outer vertices, so at least one of them is not the vertex v at which we are trying to form the radial triangulation. So choose one of the outer vertices, call it v_i, which is not v. Fix v_i and the edge which cuts it off. Note that v is in the remaining $(k-1)$-gon, so the radial triangulation at v can be formed by a series of elementary moves which do not affect the fixed edge. Then there are two cases. If v was adjacent to v_i, then the result is the desired radial triangulation. If v was not adjacent to v_i, then there exists and elementary move $\tau = v_{i-1}v_{i+1} \to v_iv$ which will result in the desired radial triangulation. □

Theorem 16. *The graph Γ_k is connected for all k.*

Proof. Assume the contrary. Consider two triangulations K and J such that there does not exist a set of elementary moves that transforms K into J. But any given triangulation can be transformed into a radial triangulation at a vertex v in the polygon by $k - 3 - d\dot{e}g(v)$ elementary moves.

Let the sequence of elementary moves that transforms K into the radial triangulation at v be $\tau_1, \ldots, \tau_r$, where $r = k - 3 - d\dot{e}g_K(v)$. Construct a similar sequence $\omega_1, \ldots, \omega_s$ in J, where $s = n - 3 - d\dot{e}g_J(v)$. Then a path from K to J is constructed by first traversing the path from K to a radial triangulation, and then from the radial triangulation to J. So the sequence of elementary moves $\tau_1, \ldots, \tau_r, \omega_s^{-1}, \omega_{s-1}^{-1}, \ldots, \omega_1^{-1}$ takes K to J. □

This also provides an explanation of the upper bound for the diameter of Γ_k found by Sleator, Tarjan, and Thurston:

Theorem 17. *The diameter of Γ_k is no greater than $2k - 10$ for $k > 12$.*

Proof. By the method above (transforming a triangulation into and then out of a radial triangulation), a triangulation K can be transformed into J by $2k - 6 - d\dot{e}g_K(()v) - d\dot{e}g_J(v)$ elementary moves for a vertex v. The sum over vertices in K of $d\dot{e}g_K(v_i)$ is $2k - 6$, so the average is $2 - \frac{6}{k}$. So the average over v_i of $d\dot{e}g_K(v_i) + d\dot{e}g_J(v_i)$ is $4 - \frac{12}{k}$. For $k > 12$, there exists a vertex v such that $d\dot{e}g_K(v) + d\dot{e}g_J(v) \geq 4$, so K is transformed into J by at most $2k - 6 - 4 = 2k - 10$ elementary moves. [2] □

But under what conditions do two triangulations realize this distance? First, a lemma in [2], whose proof I will not here present, is necessary:

Lemma 18 (Lemma 3 of [2])**.** *Let K and J be two triangulations. Let τ be an elementary move on K such that τK shares one more diagonal with J than K shares with J. Then:*

1. *There is a shortest path from K to J in Γ_k such that τ is the first elementary move made, and*
2. *If K and J share a diagonal, then that diagonal is never moved in the shortest path from K to J.*

If K and J are to be at a distance of precisely $2k - 10$ elementary moves in Γ_k, I here present the following necessary conditions:

Theorem 19. *If K and J are k-triangulations, $k \geq 14$, such that the shortest path in Γ_k from K to J has length $2k - 10$ (i.e. the Sleator, Tarjan, Thurston bound is realized), then:*

1. *K and J share no diagonals, and*
2. *For every elementary move $\tau \in EM(K)$, τK and J share no diagonals.*

Proof. It is enough to prove the second claim, since the first follows from the second.

Consider the upper bound of $2k - 10$ in [2] for the diameter of Γ_k. In fact, a weaker upper bound of $2k - 6$ shown by K. Culik and D. Wood [3] is required. This bound follows from the fact that there are $k - 3$ diagonals. Then the method described previously of moving from K to a radial triangulation and then to J will take at most $2(k - 3) = 2k - 6$ elementary moves for all $k \geq 3$.

Consider $k \geq 23$. Assume that there is an elementary move $\tau \in EM(K)$ such that τK shares a diagonal with J. By the lemma from [2], there is a shortest path from K to J beginning with τ. Perform τ to get $K' = \tau K$. Now cut each of K' and J along the shared diagonal. In both cases, the result is one j-triangulation and one $(k - j + 2)$-triangulation, where $j \geq 13$. (Throughout this proof, when a triangulation is cut into two triangulations with j and $(k - j + 2)$ vertices, I will assume $j > (k - j + 2)$ without loss of generality.) To transform the j-triangulation within K' into the j-triangulation within J requires at most $2j - 10$ elementary moves, by the upper bound in [2]. Similarly, it takes at most $2(k - j + 2) - 6$ elementary moves to transform one $(k - j + 2)$-triangulation into the other. (Here it is necessary to use the upper bound in [3] since it is possible that $k - j + 2 \leq 12$.) So the entire path from K to J will be at most $1 + 2j - 10 + 2(k - j + 2) - 6 = 2k - 11$ elementary moves. This is a contradiction, since it was assumed that K and J were $2k - 10$ elementary moves apart.

Unfortunately, for $14 \leq k \leq 22$, this proof does not work, since it cannot be guaranteed that $j > 12$. Fortunately, [2]provides calculations of the exact diameter of Γ_k for $k \leq 18$.

k	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
diameter	0	1	2	4	5	7	9	11	12	15	16	18	20	22	24	26

Again, perform τ on K such that K' shares a diagonal with J. Now there are a few cases for each value of k:

For $k = 22$, $j \geq 12$. If $j = 12$, then $k - j + 2 = 12$, and the table shows that K' and J are at most 30 elementary moves away, thus K and J are at most 31 elementary moves away. But $2(22) - 10 = 34 > 31$. And if $j > 12$, then the previous proof applies.

Similar calculations may be made for $13 \leq k < 22$, but they are straightforward, so I will not include them here. Instead, here are the calculations in table form, with the checking left to the reader:

k	14	15	16	17	18	19	20	21	22
2k - 10	18	20	22	24	26	28	30	32	34
max cutting distance	17	18	20	21	23	25	27	28	31

□

In a sense, though, this leaves us no better able to determine the shortest path from K to J in the triangulation graph. The lemma from [2] allows us to construct, under very specific conditions, such a path, but very few pairs of triangulations meet the conditions. However, I would think the following generalization of the lemma to be true:

Conjecture 20. *Let the* geodesic number $\gamma(K, J)$ *be the minimum number of elementary moves such that there exists a sequence* $\tau_1, \ldots, \tau_r$, *where* $r = \gamma(K, J)$

with the property that $\tau_r \circ \ldots \circ \tau_1 K$ *shares one more diagonal with* J *than does* K.[5] *Then there exists a shortest path from* K *to* J *which begins with* $\tau_1, \ldots, \tau_r$.

References

[1] Atsuhiro Nakamoto and Katsuhiro Ota, *Diagonal Transformations of Graphs and Dehn Twists of Surfaces*, Journal of Combinatorial Theory, Series B, Volume 70, Issue 2, July 1997, Pages 292 − 300.

[2] D. D. Sleator, R. Tarjan, W. Thurston, *Rotation Distance, Triangulations, and Hyperbolic Geometry*, Journal of the American Mathematical Society, Volume 1, Issue 3, July 1988, Pages 647 − 682.

[3] Karel Culik, Derick Wood, *A note on some tree similarity measures*, Information Processing Letters, Volume 15, Issue 1, 19 August 1982, Pages 39 − 42.

[4] Seiya Negami, *Diagonal flips in triangulations of surfaces*, Discrete Mathematics, Volume 135, Issues 1 − 3, 25 December 1994, Pages 225 − 232.

[5] Numerous conversations with Matt O'Meara and Dan Immerman, students in the REU.