The Mergelyan-Bishop Theorem

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We define the differential operator $\frac{\partial}{\partial \bar{z}}$ on infinitely differentiable functions (also called smooth or C^∞ functions) on some open set in $\mathbb{C}$ by $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$. A quick calculation shows that $\frac{\partial}{\partial \bar{z}}$ obeys the product rule. Recall that a function f is holomorphic if and only if $\frac{\partial}{\partial \bar{z}}(f) = 0$.

A function is *biholomorphic*, or an analytic isomorphism, if it is holomorphic and has holomorphic inverse. The inverse function theorem of complex analysis tells us that a holomorphic function is biholomorphic in some neighborhood of any point at which its derivative does not vanish.

The notation $G \Subset X$ means G is relatively compact in X; that is, the closure of G is compact and contained in X.

A *Riemann surface* is a (connected) one-dimensional complex manifold. An open Riemann surface is a noncompact Riemann surface.

We recall a familiar theorem from complex analysis:

Runge theorem. *Let $\Omega \subset \mathbb{C}$ be an open set and let $K \subset \Omega$ be compact. Suppose $\Omega \setminus K$ has no relatively compact connected components. Then given f holomorphic on a neighborhood on K and $\epsilon > 0$, there exists a function g holomorphic on Ω such that $\|f - g\|_K < \epsilon$.*

This theorem was proved by Runge in 1885. In 1948, Behnke and Stein extended the result to open Riemann surfaces. Their result is still generally known as the Runge theorem.

Runge approximation theorem. *Let X be an open Riemann surface and Y an open subset of X such that $X \setminus Y$ has no compact connected components. Then every holomorphic function on Y can be uniformly approximated on compact sets by functions holomorphic on X.*

For a modern proof using functional analysis, see Forster, Lectures on Riemann Surfaces.

Our main goal is the following theorem.

Mergelyan-Bishop theorem. *If X be a Riemann surface, $K \subset X$ is compact, $X \setminus K$ has no relatively compact connected components, and if f is a continuous function on K holomorphic on the interior of K, then f can be uniformly approximated on K by functions holomorphic on X.*

Remark: This theorem in the case $X = \mathbb{C}$ is a result of Mergelyan from 1954 (see Rudin, Real and Complex Analysis, Chapter 20); it was extended by Bishop to Riemann surfaces in 1958. Observe that we can assume X is an open Riemann surface, for if X is compact the theorem is entirely vacuous: either $K = X$, in which the theorem is trivial, or $K \neq X$, and $X \setminus K$ is relatively compact (because a closed subset of a compact set is compact) so the hypotheses of the theorem are not satisfied. Mergelyan's theorem is proved using techniques of measure theory, and Bishop's proof was also measure theoretical. The book Extensions of Holomorphic Functions by Jarnicki and Pflug contains (pages 86-90) a simpler proof of the Mergelyan-Bishop theorem, but their proof has a minor error. The remainder of this paper is a corrected version of Jarnicki and Pflug's proof of the Mergelyan-Bishop theorem. I make no claim to originality. I have followed their notation except when it seemed confusing or contradictory to me. I thank Raghavan Narasimhan for suggesting this topic to me and for his help.

We first collect some facts we will need in the proof.

Lemma 1: Let E be a bounded set in the complex plane and let $d\mu$ represent Lebesgue measure on the plane. Then $\int_E \frac{1}{|z|} d\mu < \infty$.

Proof: Write the integral in polar coordinates, and find $0 < R < \infty$ such that $E \subset D(0, R)$. Then

$$\int_E \frac{1}{|z|} d\mu \leq \int_0^R r dr \int_0^{2\pi} \frac{1}{r} d\theta = 2\pi R.$$

Lemma 2: Let ϕ be a compactly supported smooth function on $\mathbb{C}$, and define a function u on $\mathbb{C}$ by

$$u(z) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{\phi(\eta)}{\eta - z} d\mu,$$

where $d\mu$ represents Lebesgue measure on the complex plane.

Then u is a smooth function on $\mathbb{C}$ and

$$\frac{\partial u}{\partial \overline{z}} = \phi.$$

For a proof, see Narasimhan, Complex Analysis in One Variable, Chapter 5. The equation $\frac{\partial u}{\partial \overline{z}} = \phi$ is known as the inhomogeneous Cauchy-Riemann equation. Note that (in contrast with the situation in several variables), u does not necessarily have compact support. We will also need the following result, which is a simple consequence of the Mittag-Leffler theorem for open Riemann surfaces (see Forster, Lectures on Riemann Surfaces):

Lemma 3: Let X be an open Riemann surface and a a point in X. Then there exists a meromorphic function f on X whose only zero is at a and with $\text{ord}_a f = 1$.

The first step of the proof is to use the Mergelyan theorem to prove a local result.

Proposition 1 If X, K, and f are as before, and $a \in K$, then there exists a neighborhood V of a such that f can be uniformly approximated on $\overline{V} \cap K$ by functions holomorphic on $\overline{V} \cap K$

Proof. Choose a chart $\pi : U \to E$, where π is biholomorphic and E is a convex proper open set in $\mathbb{C}$ containing the origin. Let $V = \pi^{-1}(\frac{1}{2}E)$. Then $U \setminus \overline{V}$ is connected. Suppose $U \setminus (\overline{V} \cap K) = (U \setminus \overline{V}) \cup (U \setminus K)$ is disconnected. Then there is a component A of $U \setminus K$ contained in V and therefore relatively compact, but A will also be a component of $X \setminus K$, and by hypothesis $X \setminus K$ has no relatively compact connected components. Therefore $U \setminus (\overline{V} \cap K)$ is connected. Consequently $\pi(U \setminus (\overline{V} \cap K)) = E \setminus \pi(\overline{V} \cap K)$ is connected, and $\mathbb{C} \setminus \pi(\overline{V} \cap K)$ is also connected, and therefore has no relatively compact connected components. Therefore we can apply the Mergelyan theorem for the plane to the function $\pi^{-1} \circ f$ on the set $\pi(\overline{V} \cap K)$ to obtain a sequence (f'_j) that converges uniformly to $\pi^{-1} \circ f$ on $\pi(\overline{V} \cap K)$. Setting $f_j = \pi \circ f'_j$, we obtain functions f_j converging uniformly to f on $\overline{V} \cap K$. □

We can then complete the proof by establishing the following theorem.

Localization theorem. *Let X be an open Riemann surface, let $K \subset X$ be compact, and let f be a continuous function on K. Suppose that for every point $a \in K$, there exists a neighborhood U of a such that f can be uniformly approximated on $K \cap \overline{U}$ by functions holomorphic on $K \cap \overline{U}$. Then f can be uniformly approximated on K by functions holomorphic on X.*

We begin with a lemma.

Lemma 4: Let X be a Riemann surface, let $G_0 \Subset G \Subset X$ be open subsets of X, and choose finitely many charts $\pi_i : U_i \to E$, $1 \leq i \leq N$ such that $\{U_i\}$ is a cover of G. Then there is a positive constant C_0 with the following property: if ω is a C^∞ form of type $(0,1)$ on G and we write ω locally by $\omega|_{U_i \cap G} = \pi_i^*(\omega_j d\overline{z}_j)$, and $\|\omega_i\|_{\pi_i(U_i \cap G)} \leq C'_\omega$ for all i, then there exists a function $u \in C^\infty(G_0)$ with $\overline{\partial} u = \omega|_{G_0}$ and $\|u\|_{G_0} \leq C_0 C'_\omega$.

Corollary: Let $K \subset G \Subset X$, with K compact and G open. Choose finitely many charts $\pi_i : U_i \to E$, $1 \leq i \leq N$ such that $\{U_i\}$ is a cover of G. Then there is a positive constant C with the following property: given $\delta > 0$, if ω is a C^∞ form of type $(0,1)$ on G and we write ω locally by $\omega|_{U_i \cap G} = \pi_i^*(\omega_j d\overline{z}_j)$, and $\|\omega_i\|_{\pi_i(U_i \cap K)} \leq C_\omega$ for all i, then we can find a neighborhood G_0 of K, $G_0 \subset G$, such that there exists a function $u \in C^\infty(G_0)$ with $\overline{\partial} u = \omega$ on some neighborhood of K, and $\|u\|_K \leq CC_\omega + M\delta$ for some fixed constant M.

Remark: This corollary is necessary to fix the error in Jarnicki and Pflug's proof. When they apply lemma 3 in the proof of the localization theorem, the sets G and G_0 depend on ϵ, and therefore the constants obtained from lemma 3 also depend on ϵ. We avoid this problem by obtaining an independent estimate on K.

Proof. Choose G_0 such that $\|\omega_j\|_{\pi_j(U_j \cap G_0)} \leq 2\|\omega_j\|_{\pi_j(U_j \cap K)} + \delta$ for all j. Then choose a C^∞ function α on G such that there exists a neighborhood A of K with $\alpha(x) = 1$ for $x \in A$, $\alpha(x) = 0$ for $x \notin G_0$, and $\|\alpha\|_G = 1$. Now apply lemma 4 to the form $\alpha\omega$ to obtain $u \in C^\infty(G_0)$ with $\overline{\partial} u = \alpha\omega = \omega$ on some neighborhood of K, and $\|u\|_K \leq C_0(2C_\omega + \delta) = 2C_0 C_\omega + C_0\delta$. To finish the proof, we must establish that C_0 is independent of our choice of δ, for our choice of G_0 depended on δ. Suppose $\delta_1 > \delta_2 > 0$. We can choose (with the obvious notation) G_{0,δ_1} and G_{0,δ_2} so that $G_{0,\delta_2} \subset G_{0,\delta_1}$. Applying lemma 4 to G_{0,δ_1}, we obtain a smooth function u on G_{0,δ_1} with $\overline{\partial} u = \omega|_{G_{0,\delta_1}}$ and $\|u\|_{G_{0,\delta_1}} \leq C_{0,\delta_1} C'_\omega$. Clearly u also satisfies $\overline{\partial} u = \omega|_{G_{0,\delta_2}}$ and $\|u\|_{G_{0,\delta_2}} \leq C_{0,\delta_1} C'_\omega$, so we can set $G_{0,\delta_2} = G_{0,\delta_1}$. Therefore C_0 is independent of δ. □

Proof of Lemma 4: Choose a point $a \in \overline{G}_0$. Using lemma 3, find a meromorphic function f_a on X whose only zero is a and with $\mathrm{ord}_a f_a = 1$. Choose a neighborhood V_a of a contained in U_τ for some τ and $r > 0$ with that $f_a : V_a \to rE$ is a biholomorphism and $f_a(G\setminus V_a)\cap(rE) = \emptyset$. Multiplying f_a by a constant, we can assume that $r = 1$.

Now by the compactness of $\overline{G}_0$ we can find finitely many points $a_1, \ldots, a_k \in \overline{G}_0$ and for each a_n a meromorphic function f_{a_n} whose only zero is a_n with $\mathrm{ord}_{a_n} f_{a_n} = 1$ and a neighborhood V_n of a_n such that: (i) $f_{a_n} : V_n \to E$ is biholomorphic, (ii) $f_{a_n}(G \setminus V_n) \cap (E) = \emptyset$, (iii) $G_0 \subset \cup_{n=1}^k V_n \subset G$, and (iv) V_n is contained in one of the sets U_i.

Now choose a partition of unity φ_n on $\overline{G}_0$ with respect to the open cover $(V_n)_{n=1}^k$. Define ω'_n on V_n by $\omega'_n = \varphi_n \omega$. Let $\tilde{z_n}$ be the complex variable on the coordinate neighborhood (V_n, f_{a_n}), and find functions $\tilde{\omega'_n}$ such that $\omega'_n = f^*_{a_n}(\tilde{\omega'_n})d\overline{\tilde{z_n}}$.

Define

$$g_n(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\tilde{\omega'_n}(\eta}{\eta - z} d\mu,$$

where $d\mu$ represents Lebesgue measure on the complex plane. Then by lemma 1, $\frac{\partial g_n}{\partial \bar{z}} = \tilde{\omega'_n}$. We have

$$\begin{aligned}
\frac{\|g_n\|_{\mathbb{C}}}{\|\tilde{\omega'_n}\|_{\mathbb{C}}} &= \frac{1}{\pi}\mathrm{sup}_{z\in\mathbb{C}} \left| \int_E \frac{\tilde{\omega'_n}(\eta)}{\|\tilde{\omega'_n}\|_{\mathbb{C}}(\eta - z)} d\mu \right| \\
&\leq \frac{1}{\pi}\mathrm{sup}_{z\in\mathbb{C}} \int_E \left| \frac{\tilde{\omega'_n}(\eta)}{\|\tilde{\omega'_n}\|_{\mathbb{C}}(\eta - z)} \right| d\mu \\
&\leq \frac{1}{\pi}\int_D \frac{1}{|z|} d\mu
\end{aligned}$$

where D is a fixed bounded open set. Therefore by lemma 1, we have $\|g_n\|_{\mathbb{C}} \leq C'_0 \|\tilde{\omega'_n}\|_{\mathbb{C}}$ where C'_0 is a fixed positive constant independent of ω and of n.

Note that as $\tilde{\omega'_n}(\eta) = 0$ if $\eta \notin E$, we have $\frac{\partial g_n}{\partial \bar{z}} = 0$ on $\mathbb{C} \setminus E$, so g_n is holomorphic on $\mathbb{C} \setminus E$. We just showed that g_n is bounded, so we can consider g_n as a function defined on the Riemann sphere and holomorphic at infinity.

Now define $\widehat{g_n} = g_n \circ f_{a_n}$. Clearly each $\widehat{g_n}$ is a smooth function on X. As f_{a_n} is holomorphic on G_0, $f_{a_n}(G \setminus V_n) \cap (E) = \emptyset$, and g_n is holomorphic on $\mathbb{C} \setminus E$, $\widehat{g_n}$ is holomorphic on $G_0 \setminus V_n$. We can calculate that on G_0

$$\overline{\partial}\widehat{g_n} = \overline{\partial}(g_n \circ f_{a_n}) = f^*_{a_n}(\overline{\partial} g_n) = f^*_{a_n}(\tilde{\omega'_n})d\overline{\tilde{z_n}} = \omega'_n.$$

Now let $u = \sum_{n=1}^{N} \widehat{g_n}$. Clearly u is smooth, and on G_0

$$\overline{\partial} u = \sum_{n=1}^{N} \overline{\partial} g_n = \sum_{n=1}^{N} \omega_n' = \sum_{n=1}^{N} \varphi_n \omega = \omega$$

and

$$\|u\|_{G_0} \leq \sum_{n=1}^{N} \|g_n\|_{G_0} \leq C_0 C_\omega',$$

for some fixed positive constant C_0, where the last inequality comes from changing coordinates between the π_j and the f_{a_n} and the fact that $\|g_n\|_{\mathbb{C}} \leq C_0' \|\widetilde{\omega_n'}\|_{\mathbb{C}}$. This completes the proof of the lemma.

We can now finish the proof of the localization theorem. By the compactness of K, we can choose a finite number of charts $\pi_j : U_j' \to E$, π_j biholomorphic, $E \subset \mathbb{C}$ convex and bounded, $1 \leq j \leq N$, in such a way that if we define $U_j = \pi_j^{-1}(\frac{1}{2}E)$, we have $K \subset \cup_{j=1}^{N} U_j$ and so that f can be uniformly approximated on $K \cap \overline{U}_j$ by functions holomorphic on $K \cap \overline{U}_j$ for all $j = 1, \ldots, N$.

Now let $\epsilon > 0$, and let f_j be holomorphic on $K \cap \overline{U}_j$ with $\|f - f_j\|_{K \cap \overline{U}_j} < \frac{\epsilon}{2}$. As an easy application of the fact that power series converge on disks, we see that there exist open sets Ω_j, $K \cap \overline{U}_j \subset \Omega_j \Subset U_j'$ with f_j holomorphic on Ω_j. Now be continuity, choose open sets Ξ_j, $K \cap \overline{U}_j \subset \Xi_j \subset \Omega_j$ such that $\|f - f_j\|_{\Xi_j} < \epsilon$ for all $1 \leq j \leq N$.

Now choose an open set G with $K \subset G \Subset \cup_{j=1}^{N} U_j$. Choose a partition of unity φ_j for G with respect to the cover $(U_j)_{j=1}^{N}$.

If $1 \leq j \leq N$, $1 \leq k \leq N$, define

$$h_{j,k}(z) = \begin{cases} \varphi_j(z)(f_j(z) - f_k(z)) & z \in \Xi_j \cap \Xi_k \\ 0 & z \in \Xi_k \setminus \overline{U}_j \end{cases}$$

Then $h_{j,k}$ is a C^∞ function on its domain, which is $(\Xi_j \cap \Xi_k) \cup (\Xi_k \setminus \overline{U}_j)$; we will denote this set by $\Xi_{j,k}'$. Now let $\Xi_k' = \cap_{j=1}^{N} \Xi_{j,k}'$. If $z \in K \cap \overline{U}_k$ and $1 \leq j \leq N$, then $z \in \Xi_k$ and if $z \notin \Xi_j$, then $z \notin \overline{U}_j$ (because $\overline{U}_j \cap K \subset \Xi_j$. Therefore $z \in \Xi_{j,k}'$, and $K \cap \overline{U}_k \subset \Xi_k'$, so $(\Xi_k')_{k=1}^{N}$ is a cover of K.

Next we define $h_k = \sum_{j=1}^{N} h_{j,k}$; clearly the domain of h_k is Ξ_k' and h_k is C^∞. Let $z \in \Xi_k' \cap K$. Then $|h_k(z)| = |\sum \varphi_j(z)(f_j(z) - f_k(z)| \leq \sum |\varphi_j(z)(f_j(z) - f_k(z)|$, where the sums are over those $j \in 1, \ldots, N$ with $z \in U_j$. Recall that $\|f_j - f\|_{K \cap \Xi_j} < \epsilon$. Therefore $\|f_j - f_k\|_{K \cap \Xi_j \cap \Xi_k} < 2\epsilon$, so using the fact that φ is a partition of unity, we have $\|h_k\|_{K \cap \Xi_k'} < 2\epsilon$.

Consider the form $\overline{\partial} h_k|_{K\cap\Xi'_k} = \pi^*\left(\frac{\partial(h_k\circ\pi_k^{-1})}{\partial\overline{z}_k} d\overline{z}_k\right)$. Because $\frac{\partial}{\partial\overline{z}}$ obeys the product rule, we can find, using the fact that Ξ'_k is relatively compact and that $f_j - f_k$ is holomorphic, a constant C^* independent of j and ϵ such that

$$\sup_{x\in K\cap\Xi'_k}\left|\frac{\partial(h_k\circ\pi_k^{-1})}{\partial\overline{z}_k}(\pi_k(x))\right| \leq C^* 2\epsilon.$$

By continuity, find an open set G_0, $K \subset G_0 \Subset G \cap (\cup_{k=1}^N \Xi'_k)$ such that

$$\sup_{x\in G_0\cap\Xi'_k}\left|\frac{\partial(h_k\circ\pi_k^{-1})}{\partial\overline{z}_k}(\pi_k(x))\right| \leq C^* 3\epsilon.$$

A quick calculation tells us that $h_k - h_j = f_j - f_k$ on $G\cap\Xi'_k\cap\Xi'_j$, so $h_k - h_j$ if holomorphic on $G\cap\Xi'_k\cap\Xi'_j$, so $\overline{\partial} h_k = \overline{\partial} h_j$ on $G\cap\Xi'_k\cap\Xi'_j$, so we can piece together a $C^\infty_{(0,1)}$ form α on $G\cap(\cup_{k=1}^N \Xi'_k)$ with $\alpha = -\overline{\partial} h_k$ on $G\cap\Xi'_k$.

Let $\delta > 0$. By the corollary to lemma 4, there exists a smooth function u on G_0 with $\overline{\partial} u = \alpha$ on some neighborhood B of K and $\|u\|_K < CC^* 3\epsilon + M\delta$ for fixed constants C, C^*, and M.

Set $g_k = u + h_k$ on $\Xi'_k\cap G_0$, then $\overline{\partial} g_k = \overline{\partial} u + \overline{\partial} h_k = 0$ so g_k is holomorphic, and $\|g_k\|_{\Xi'_k\cap K} < 2\epsilon + CC^* 3\epsilon + M\delta$. Now $g_k - g_j = h_k - h_j = f_j - f_k$ on $\Xi'_k\cap\Xi'_j\cap G_0$, so $g_k + f_k = g_j + f_j$ on $\Xi'_k\cap\Xi'_j\cap G_0$, so we can piece together a holomorphic function F on G such that $F|_{\Xi'_k\cap G} = f_k + g_k$. Furthermore, for $x\in\Xi'_k\cap K$ we have

$$|f(x) - F(x)| = |f(x) - f_k(x) - g_k(x)|$$

$$\leq |f(x) - f_k(x)| + |g_k(x)| < \epsilon + 2\epsilon + CC^* 3\epsilon + M\delta.$$

This proves that f can be uniformly approximated on K by holomorphic functions on an open neighborhood of K. By the Runge theorem, f can be uniformly approximated on K by holomorphic functions on X. This completes the proof of the localization theorem.