The Number of Elementary Moves Required to Begin to Swap Triangulations

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An elementary move is a process which takes you from one triangulation to another. The process is as follows: start with a triangulation, remove a diagonal, thus forming a quadrilateral, and then replace the opposite diagonal in the quadrilateral, which brings you back to a triangulation. Given two triangulations of a regular polygon T_1 and T_2, one might wonder how many elementary moves one must apply to T_1, in order to make the resulting figure share one more diagonal with T_2. I will define this as the geodesic number of T_1 and T_2 or $\gamma(T_1, T_2)$ for short. In this paper, I will show that the smallest n possible, such that two n-gons, T_1 and T_2 have $\gamma(T_1, T_2) = k$ is $n = 2k + 2$

Background: Sleater, Tarjan, and Thurston showed in 1988 that if $\gamma(T_1, T_2) = 1$, then the corresponding elementary move was the start of the shortest move from T_1 to T_2 [1]. My friend Andy Eisenberg conjectured that if $\gamma(T_1, T_2) = k$ then the corresponding series of k elementary moves was the start of the shortest move from (T_1 to T_2) [2]. If this was true, he would have an algorithmic process for calculating how many elementary moves it would take to get from any triangulation to another. He was curious about how fast $\gamma(T_1, T_2)$ could grow in relation to the size of the triangulation. Hence this paper.

1 Proof that n is $\geq 2k + 2$

Elementary Fact 1.1: Any triangulation has at least two diagonals whose endpoints have only one point in between them.

Defintion 1.2: I call the middle point, which connects to 0 diagonals, an outer vertex (See Figure 1)

Lemma 1.3: In order for $\gamma(T_1, T_2)$ to be m, it is necessary that the corresponding point in T_1 to an outer vertex in T_2 must have at least m diagonals coming

[1] D. Sleator, R. Tarjan, and W. Thurston, Rotation distance, triangulations, and hyperbolic geometry, J. Amer. Math. Soc. 1 (1988), 647-681

[2] personal communication

out of it

Proof: During each elementary move, one may remove one diagonal from a point. Therefore, if there were less than m diagonals coming out of the point corresponding to the outer vertex, it would take less than m moves for the two figures to share a diagonal

Elementary Fact 1.4: a triangulation of a j-gon will contain j-3 diagonals

Claim 1.5: given $\gamma(T_1, T_2) = k, n$ must be $\geq 2k + 2$

Proof: Consider two triangulations, T_1 and T_2 on an n-gon. By Lemma 1.3 In order for $\gamma(T_1, T_2) = k$, T_1 must have at least k diagonals coming out of each of the two points corresponding to the outer vertices in T_2. Because these points can share only one diagonal in common, this leads to a total of at least 2k-1 diagonals. By Elementary Fact 1.4, this means that $n \geq 2k + 2$

2 Proof that n is $\leq 2k + 2$

Proof 2.1: by construction

Construct the two triangulations as follows:

(See Figure 2) In T_1 connect point 2 to all of the points between k+2 and 2k+2. Connect point k+2 to all of the points 3 through k. In T_2 connect point 1 all of the points between 2 and k+1. Connect point k+1 to all of the points between k+3 and 2k+2.

Claim 2.2: $\gamma(T_1, T_2) = k$

Proof: Any of the lines passing through point 2 in T_1 intersect all of the lines passing though the point 1 in T_2 so it would take at least k elementary moves to connect a diagonal in T_2 to point 2. Likewise, any of the lines passing through point k+2 in T_2 intersect all of the lines passing through point k+1 in T_2 so it would take at least k elementary moves to connect a diagonal in T_2 to point k+2. But all of the diagonals in T_1 are connected either to point 2 or k+2, so it would take at least k elementary moves to make a line in T_2 match up with one in T_1.