MATHEMATICAL PHYSICS
A SURVEY OF GAUGE THEORIES AND SYMPLECTIC TOPOLOGY

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Physics has always served as an inspiration for mathematical growth, while mathematics has always allowed for a simpler and deeper understanding of physics. Today is no different. Two fundimental branches of physics: field theory and classical mechanics have been rigorously formalized in the mathematical branches of fiber bundle theory and symplectic topology. This paper aims to desrcibe these physical processes within the framework of these mathematical disciplines.

1. Field Theory and Bundle Theory

The behavior of a physical field is completely determined by two criteria:

(1) The *geometry* of the space in which the field exists
(2) The *action* of the field, which it minimizes.

By knowing both the geometry and the action of a field, one may find the equations of motion of the field, and hence know all there is to know about this field. [Note: In this paper, we only discuss non-interacting fields. We assume space is filled with our field and void of all else.] In the following, we will describe the geometry of vector bundles and how this determines physical properties of a field. We shall not discuss action here, however this notion is of great significance, and leads into Gauge Theory [for more on Gauge Theory, see [2]].

Let M be a smooth manifold. A *scalar field* ϕ on M is a physically measurable property. Our measurements take values in a field k, ($\mathbb{R}$ or $\mathbb{C}$). Hence a scalar field is a map $\phi : M \to k$. For example, the surface tempurature of the earth is a scalar field $T : S^2 \to \mathbb{R}$. Another example is that of a quantum mechanical state function ψ that satisfies Schroedinger's equation, in which case $\psi : M \to \mathbb{C}$. If $\phi_1, \phi_2, \cdots, \phi_n$ are n noninteracting scalar fields on M with values in F, then an *n-scalar field* $\Phi = (\phi_1, \phi_2, \cdots, \phi_n)$ is a map

$$\Phi : M \to k^n$$

We shall call the pair (M, Φ) a *physical system.* Associated to any physical system is a *potential field* A, (also known as a *gauge field*) which in some sense measures what it means for Φ to change from point to point in M. In so doing, A is realized as a map

$$A : TM \to gl_n(k)$$

Two example of gauge fields are the electric and gravitational potentials. Now the difference in potential between two points gives rise to a force. Thus, if D is the apporopriate differential operator, the field strength F associated to Φ is given by $DA = F$. It should be noted that the gauge potential A is additional information one must specify, it does not follow from (M, Φ). However, upon choosing a specific (M, Φ, A) the geometry of system determines the field strength F.

Definition 1.1. Let E, M, and F be smooth manifolds, and G a Lie group acting continuously and faithfully on F. A *smooth fiber bundle* ξ over the base space M with total space E, fiber F, and structure group G is a surjective smooth map $\pi : M \to B$ called the bundle projection, together with a maximal bundle atlas $\mathcal{A} = \{(\pi^{-1}(U_\alpha), \Phi_\alpha\}_{\alpha \in A}$ satisfying

(1) $\{U_\alpha\}_{\alpha \in A}$ is an open cover of M.
(2) $\Phi_\alpha = (\pi, \phi_\alpha)$ where the map $\phi_\alpha : \pi^{-1}(U_\alpha) \to F$ is smooth and

$$(\pi, \phi_\alpha) : \pi^{-1}(U_\alpha) \to U_\alpha \times F; \quad (\pi, \phi_\alpha)(p) = (\pi(p), \phi_\alpha(p))$$

is a diffeomorphism
(3) Given $\alpha, \beta \in A$, there is a smooth map $f_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ such that, given $p \in U_\alpha \cap U_\beta$,

$$f_{\alpha\beta}(p) : F \to F; \quad f_{\alpha\beta}(p) = \phi_\beta \circ (\phi_\alpha|_{\pi^{-1}(p)})^{-1}$$

or equivalenty

$$\phi_\beta|_{\pi^{-1}(U_\alpha \cap U_\beta)} = (f_{\alpha\beta} \circ \pi) \cdot \phi_\alpha|_{\pi^{-1}(U_\alpha \cap U_\beta)}.$$

Definition 1.2. A *vector bundle of dimension* n is a fiber bundle $\xi = (E, M, V, \pi)$ with fiber an n-dimensional vector space V and structure group a subgroup of $GL(V)$. A *principle G-bundle* is a fiber bundle $\xi = (E, M, G, \pi)$ whose fiber and structure group is G which acts freely and G-equivariently on E.

Given the above motivation, we can see that if M is spacetime, and Φ is an n-scalar field on M with values in V, then this physical system is realized by an n-dimensional vector bundle $\xi = (E, M, V, \pi)$.

Definition 1.3. A *smooth section* of a fiber bundle is a smooth map $s : M \to E$ such that $\pi \circ s = \mathrm{id}_M$. We denote the space of smooth sections as $\Omega^0(\xi)$. A vector field on M is a smooth section of the tangent bundle τ_M.

A specific configuration of Φ on M is simply a section into E. Given two vector bundles $\xi_1 \equiv \pi_1 : E_1 \to M$ and $\xi_2 \equiv \pi_2 : E_2 \to M$, we may define new vector bundles over M fiberwise:

(1) $\xi_1 \oplus \xi_2$
(2) $\xi_1 \otimes \xi_2$
(3) ξ_1^*
(4) $\mathrm{Hom}_k(\xi_1, \xi_2)$

One can check that the total spaces are topologized in the obvious way.

Definition 1.4. Let $\xi = (E, M, V, \pi)$ be a vector bundle. A *Euclidian metric* on ξ is a smooth section s into the bundle $(\xi \otimes \xi)^*$. If ξ is the tangent bundle τ, then this is a *Riemannian metric.*

Theorem 1.5. *$\xi \equiv \pi : E \to M$ a vector bundle. Then ξ admits a Euclidian metric.*

Proof: By assumption, M is a smooth manifold, so we may pick a locally finite atlas $\mathcal{A} \equiv \{U_\alpha, \phi_\alpha\}_{\alpha \in A}$ which is also a local trivialization of ξ. Let $\{p_\alpha\}_{\alpha \in A}$ be a partition of unity subordinate to $\{U_\alpha\}_{\alpha \in A}$. Let g_α be the euclidean metric on $\phi_\alpha(U_\alpha) \subset \mathbb{R}^n$. Then we pull back along $\phi_\alpha \circ \pi$ to get a $(0, 2)$-tensor field $(\phi_\alpha \circ \pi)^* g_\alpha$ on $\pi^{-1}(U_\alpha)$. Define the map

$$s : M \to (E \otimes E)^* \qquad s(x) = \sum_{\alpha \in A} p_\alpha(x)(\phi_\alpha \circ \pi)^* g_\alpha$$

It immediately follows that s is our desired section.

Corollary 1.6. *Every smooth manifold admits a Riemannian metric.*

Definition 1.7. A *Riemannian manifold* is a pair (M, g) where M is a smooth manifold and g is a Riemannian metric on M.

A given manifold M may admit multiple Riemannian metrics, and each would correspond to a different Riemannian manifold. While the standard metric g_0 on $\mathbb{R}^4$ is given by

$$g_0(\mathbf{v}, \mathbf{w}) = v_1w_1 + v_2w_2 + v_3w_3 + v_4w_4$$

physicists usually work with the Minkowski metric η on $\mathbb{R}^4$, which is given by

$$\eta(\mathbf{v}, \mathbf{w}) = -v_1w_1 + v_2w_2 + v_3w_3 + v_4w_4$$

Although this topic will not be pursued in this paper, there is much to be said about Riemannian manifolds and their relationships with physics.

Let (M, Φ) be a physical system. Given $p, q \in M, p \neq q$, it is natural to ask how one compares $\Phi(p)$ and $\Phi(q)$? To say that two fibers are isomorphic is certainly not the same as saying that two fibers are equal. So physically and mathematically, we just do not have any reasonable way to resolve this issue as is. This motivates the notion of a *connection*. There are three equivalent ways to define a connection on vector bundles, and in this treatment, we begin with the most geometric and derive the others.

Consider the vector bundle $\xi = (E, M, V, \pi)$. Pick some point $v \in E$. Now, sitting at v there are just two directions to go: either along the fiber, or across into a different fiber. Moving within the fiber changes the vector, so by only moving *across* fibers, the value of the original vector is retained. To be a bit more precise, π induces the map $\pi_* : TE \to TM$ of tangent bundles, so $\pi_{*v} : T_vE \to T_{\pi(v)}M$ is linear. A vector whose image is zero corresponds to motion vertically along the fiber. Let $\mathcal{V}_v = \ker(\pi_{*v})$ and $\mathcal{H}_v = \text{Im}(\pi_{*v})$. Then $T_vE \cong \mathcal{H}_v \oplus \mathcal{V}_v$, where $\mathcal{H}_v$ represents horizontal movement across fibers.

Definition 1.8. A *k-dimensional distribution* Δ on M is a smooth field of k-dimensional subspaces of TM.

Definition 1.9. A connection on ξ is a distribution $\mathcal{H}$ satisfying

(1) For each $v \in E, \pi_* : \mathcal{H}_v \to T_{\pi(v)}M$ is an isomorphism
(2) For $a \in \mathbb{R}$, let μ_a be multiplication by a. Then $\mu_{a*}\mathcal{H}_v = \mathcal{H}_{av}$.

Based on the above observations and this definition, we see that $TE \cong \mathcal{H} \oplus \mathcal{V}$. Hence any vector field X on TE can be decomposed as $X = X^h + X^v$ where $X^h \in \mathcal{H}$ and $X^v \in \mathcal{V}$. A path

$\gamma : I \to E$ is horizontal if $\dot{\gamma}(t)$ is a horzontal vector field along γ. Ideally, if one were to parallel transport a vector along a horizontal loop, one would end up with the same vector you started with. In this way, the level sets of parallel translation along horizontal paths would sweep out a smooth submanifold of E. When we view our connection as being the potential of some force, it follows that if all vectors are always parallel translated back to themselves, then the potential is constant. Unfortunantely, this is not always the case.

Definition 1.10. A distribution is *integrable* if $X, Y \in \Delta$ implies $[X, Y] \in \Delta$. A submanifold $i : N \hookrightarrow M$ is integrable if $i_* : TN \hookrightarrow TM$ is an integrable distribution.

Definition 1.11. A *k-dimensional foliation* of M is a partitioning of M into k-dimensional submanifolds called leaves satisfying:

(1) The collection of tangent spaces to the leaves form a distribution Δ on M
(2) If $N \subset M$ is an integral submanifold with distribution Δ, then N is contained in one of the leaves.

Theorem 1.12. *(Frobenius) A distribution is integrable iff it is induced by a folliation.*

The proof of this theorem is rather technical, and does not contribute to the content of this paper, so the reader is referred to either [5] or [6]. Now we can say that ideally, a connection defines a folliation of the total space E, or equivalently by the Frobenius theorem, $\mathcal{H}$ is an integrable distribution. Since $\mathcal{H}$ is not always integrable, we can at least measure how far away it is from being integrable. To this end we construct a sort of directional derivative for sections.

Let $p_2 : TE = \mathcal{H} \oplus \mathcal{V} \to \mathcal{V}$ be the canonical projection. The pullback of E along π yields $\pi^*E = \{(v_1, v_2) \in E \times E | \pi(v_1) = \pi(v_2)\}$. Let

$$\pi_2 : \pi^*E \to E; \qquad \pi_2(v_1, v_2) = v_2.$$

We may consider a pair $(v_1, v_2) \in \pi^*E$ as a fixed point in the fiber v_1, and a direction in the fiber $v_2 - v_1$. In this way we see that this pullback bundle is precisely the verticle bundle. To be precise, consider the map

$$\eta : \pi^*E \to \mathcal{V}; \qquad \eta(v_1,v_2) = \frac{d}{dt}(v_1 + tv_2)|_{t=0}$$

For a given $v \in E$, we see that $\eta_v : \pi^* E_v \to \mathcal{V}_v$ is linear, and furthermore, it follows that $\ker \eta_v = 0$. By dimensionality considerations it follows that η_v is an isomorphism, and hence η is an equivalence of bundles. Now we define the *connection map*

$$\kappa : TE \to E; \qquad \kappa(X) = \pi_2 \circ \eta^{-1} \circ p_2(X).$$

Given a vector $v \in E$, and a vector $X_v \in T_v E$, we constucted κ to give the amount v changes in the direction of X_v. With this we define a notion of a directional derivative on a vector bundle, which we call the *covarient derivitive.*

$$\nabla : \Omega^0(\xi) \to Hom(\Omega^0(\tau_M), \Omega^0(\xi)) \quad \nabla(s) = \kappa \circ s_*$$

Fixing a vector field X on M, and evalutaing at it gives

$$\nabla_X : \Omega^0(\xi) \to \Omega^0(\xi) \quad \nabla_X(s) = \kappa \circ s_*(X)$$

Which measures how much a section changes in the direction of X. In this way, ∇ measures how far a section is from being horizontally transported, if X is horizontal $\nabla_X = 0$. Thus ∇ is precisely the tool by which we can measure how far $\mathcal{H}$ is from being integral. The existance of the covarient derivative ∇ on ξ is equivalent to the existance of a specific connection $\mathcal{H}$ and for this reason ∇ is sometimes called the connection on ξ.

As $s \in \Omega^0(\xi)$ is a $\Omega^0(M)$-module, it immediately follows that

$$\Omega^0(\xi) = \Omega^0(M) \otimes_{\Omega^0(M)} \Omega^0(\xi)$$

We can then generalize concept and make the following definition:

$$\Omega^k(\xi) \equiv \Omega^k(M) \otimes_{\Omega^0(M)} \Omega^0(\xi)$$

It follows that $\nabla \in \Omega^1(\xi)$. So, if $U \subset M$ is a trivializing neighborhood, we may pick n sections $s_1, , s_n$ such that $s_1(p), , s_n(p)$ are a basis for V_p for each $p \in M$. We call this a frame on U. [In fact this is just a section into the *frame bundle* of M which is the principle $GL_n(V)$-bundle associated to ξ.] Hence on U

$$\nabla(s_i) = \sum_{j=1}^{n} A_{ij} \otimes s_j$$

where $A_{ij} \in \Omega^1(M)$ for each i and j, or in other words, A is a gl_n-valued one-form on U. Hence the connection ∇ on U is given by A. [Of course, if we were to say, pick a different section into the frame bundle, the resulting A' would be different than A. However, A and A' would be related by some action of $GL_n(V)$. This is one of the key ideas of Gauge theory, which will not be discussed in this paper.]

Define the map

$$F^\nabla : \Omega^0(\xi) \to \Omega^2(\xi)$$

where, upon picking $X, Y \in \Omega^0(TM)$,

$$F^\nabla_{XY}(s) \equiv F_\nabla(X, Y, s) = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s$$

Theorem 1.13. *$F^\nabla = 0$ iff $\mathcal{H}$ is integrable*

Once again, proof of this theorem is rather technical, and does not contribute to the content of this paper, so the reader is referred to [5]. F^∇ measures the amount by which $\mathcal{H}$ fails to be integrable. If we choose a frame on $U \subset M$, then

$$F^\nabla(s_i) = \sum_{j=1}^{n} F_{ij} \otimes s_j$$

where $F_{ij} \in \Omega^2(M)$ for each i and j, or in other words, F is a gl_n-valued two-form on U. F is the *curvature* of U associated with the connection A. [A different choice of frame will change F to F', but these two are related by an action of $GL_n(V)$. This is key in Gauge theory, but will not be discussed further here.]

Define the differential operator

$$D : \Omega^k(M) \otimes gl_n(V) \to \Omega^{k+1}(M) \otimes gl_n(V) \qquad D\omega = d\omega + A \wedge \omega$$

where d is the standard differential from deRham cohomology.

Theorem 1.14. $F = DA$

The reader is referred to [2] or [5] for more discussion of the differential operator D and for the proof of the above relation. To conclude, if we are given a physical system (M, Φ), then we may choose a connection $\mathcal{H}$ on the vector bundle $\xi = (E, M, V, \pi)$ which give the connection 1-form A. This 1-form is the potential corresponding to Φ, and the derivative $F = DA$, which is the curvature of the vector bundle, gives the field strength corresponding to Φ.

2. Classical Mechanics and Symplectic Geometry

Classical mechanics is concerned with the non-relativistic dynamical behavior of a system. So first we must understand how to describe a system. Let M be a smooth manifold and consider some physical system occuring in M.

Definition 2.1. A system has n degrees of freedom if the position of the system is completely described by n numbers and n is minimal.

While a system's position at a fixed time is given by a point $q = (q_1, q_2, \cdots q_n) \in \mathbb{R}^n$, this does not give all the information needed to describe the state of the system. A sufficient collection of further data is the generalized momenta $p = (p_1, p_2, \cdots p_n) \in \mathbb{R}^n$ associated to the generalized coordinates. It follows that the states of a mechanical system is completely specified by a point $z = (q_1, q_2, \cdots q_n, p_1, p_2, \cdots p_n) \in \mathbb{R}^{2n}$, which is called *phase space* T^*M. While the state of the system in now entirely specified, there is still not enough information to understand how it will evolve with time. Physically, a system will evolve in time in a way that minimizes energy. Energy is simply a smooth scalar field on M, which we call the Hamiltonian H. Since H is minimized, variational calculations yields the solutions:

$$\frac{dp_i}{dt} = \frac{dH}{dq_i}$$

$$\frac{dq_i}{dt} = -\frac{dH}{dp_i}$$

These equations define a flow on T^*M that specify how the state of a system evolves over time.

With this as motivation, physics [not to mention mathematics] is very interested in understanding possible phase spaces and possible Hamiltonian flows on them with the greatest amount of rigor and generality. This is just the beginning of symplectic topology.

Definition 2.2. A symplectic structure on M is given by a closed and non-degenerate 2-form ω. A symplectic manifold is a pair (M, ω).

Just from these properties we see a very striking fact about symplectic manifolds, that being:

Proposition 2.3. *M a smooth manifold. If M admits a symplectic structure, then M is an even dimensional manifold.*

Proof: For each point $p \in M^m$, ω_p is a nondegenerate, 2-form on T_pM. For the sake of this proof, we shall call the tangemt space at p V_1. Pick some $x_1 \in V_1, x_1 \neq 0$. Nondegeneracy implies that the map $i_{x_1}\omega_p : V_1 \to \mathbb{R}$ is onto and hence $\dim_{\mathbb{R}}\ker(i_{x_1}\omega_p) = m-1$. The subspace of vectors with nonzero image is one dimensional, and hence there exists a $y_1 \in V_1$ such that $\omega_p(x_1, y_1) = 1$ and $V_1 \cong \ker(i_{x_1}\omega_p) \oplus y_1\mathbb{R}$. Let $V_2 = V_1 \setminus (x_1\mathbb{R} \oplus y_1\mathbb{R})$. It follows that $\omega_p(x_1, v) = 0 = \omega_p(y_1, v)$ for all $v \in V_2$ and consequently ω_p is a nondegenerate 2-form on V_2. Continuing this process n times, where $m = 2n + \epsilon$, $\epsilon = 0$ or 1 yields

$$T_pM \cong x_1\mathbb{R} \oplus y_1\mathbb{R} \oplus \cdots \oplus x_n\mathbb{R} \oplus y_n\mathbb{R} \oplus V_{n+1}$$

where $\dim V_{n+1} = \epsilon$, $\omega_p(x_i, x_j) = 0 = \omega_p(y_i, y_j)$ and $\omega_p(x_i, y_j) = \delta_{ij}$. Pick $v \in V_{n+1}$. By construction, $\omega_p(v, w) = 0$ for all $w \in V_1 \setminus V_{n+1}$ and skew-symmetry implies that $\omega_p(v, w) = 0$ for all $w \in V_{n+1}$. Thus $\omega_p(v, w) = 0$ for all $w \in V_1$ and nondegeneracy implies that $v = 0$. Hence $\epsilon = 0$.

The definition of a Riemannian metric and a symplectic form are very similar, but already we see a striking difference between Riemannian geometry and symplectic geometry. While every smooth manifold admits a Riemannian metric, we see immediately that no odd dimensional manifold admits a symplectic structure, and in fact, it is a nontrivial question as to when a manifold will admit a symplectic structure.

Examples 2.4. (1) $(\mathbb{R}^{2n}, \omega_0)$ where $\omega_0 = \Sigma_{i=1}^n dx_i \wedge dy_i$

(2) (S^2, τ) where we define pointwise $\tau(v, w)_p =< p, v \times w >$

(3) Let M be any smooth n-manifold. Then (T^*M, ω) is a symplectic manifold where T^*M is the cotangent bundle, and $\omega = \sum_{i=1}^n dx_i + dy_i$ where $\{x_i\}$ are the coordinates on M and $\{y_i\}$ are the coordinates on the fiber.

Proposition 2.5. *M a smooth manifold with a smooth family of symplectic forms ω_t with exact derivative $\frac{d}{dt}\omega_t = d\sigma_t$. Then there exists a smooth family of diffeomorphisms ψ_t such that $\psi^*\omega_t = \omega_0$.*

Proof: A family of diffeomorphisms ψ_t exists if and only if they may be realized as a the flow of a smooth family of vector fields X_t, i.e.

$$\frac{d}{dt}\psi_t = X_t \circ \psi_t$$

. If such a family exists, then, taking the derivative of $\psi^*\omega_t = \omega_0$ yields.

$$
\begin{aligned}
0 &= \frac{d}{dt}\omega_0 \\
&= \frac{d}{dt}(\psi_t^*\omega_t) \\
&= \lim_{h\to 0} \frac{\psi_{t+h}^*\omega_{t+h} - \psi_t^*\omega_t}{h} \\
&= \lim_{h\to 0} \left[\frac{\psi_{t+h}^*\omega_{t+h} - \psi_{t+h}^*\omega_t}{h} + \frac{\psi_{t+h}^*\omega_t - \psi_t^*\omega_t}{h}\right] \\
&= \psi_t^*(\frac{d}{dt}\omega_t) + \psi_t^*(\mathcal{L}_{X_t}\omega_t) \\
&= \psi_t^*\left[\frac{d}{dt}\omega_t + (d\circ i_{X_t} + i_{X_t}\circ d)\omega_t\right] \\
&= \psi_t^*\left[\frac{d}{dt}\omega_t + d(i_{X_t}\omega_t) + i_{X_t}(d\omega_t)\right] \\
&= \psi_t^*\left[d\sigma_t + d(i_{X_t}\omega_t)\right] \\
&= d\left[\psi_t^*\ \sigma_t + i_{X_t}\omega_t\right]
\end{aligned}
$$

Now the theory of differential equations [which shall not be discussed here, see [5]] gives the existence of a family of vector fields satisfying $0 = \psi_t^*\ \sigma_t + i_{X_t}\omega_t$. The nondegeneracy of ω_t implies that this solution is unique. Hence we have the existence of the desired family of diffeomorphisms.

Theorem 2.6. *Let M be a 2n-dimensional smooth manifold and $Q \subset M$ a compact submanifold. Suppose that $\omega_0, \omega_1 \in \Omega^2(M)$ are closed 2-forms such that at each point q of Q the forms ω_0 and ω_1 are equal and nondegenerate on T_qM. Then there exist open neighborhoods $\mathcal{N}_0$ and $\mathcal{N}_1$ of Q and a diffeomorphism $\psi : N_0 \to N_1$ such that*

$$\psi|_Q = id, \quad \psi^*\omega_1 = \omega_0$$

Sketch of Proof: The smoothness of ω_0 and ω_1 on M assures us that we may choose a small enough open neighborhood $\mathcal{N}_0$ of Q in which ω_0 and ω_1 are nondegenerate. Let $\tau = \omega_1 - \omega_0$ and define $\omega_t = \omega_0 + t\tau$. Hence ω_t is a smooth family of symplectic forms on $\mathcal{N}_0$ such that $\frac{d}{dt}\omega_t = d\tau_t$. If τ is closed, then by the above proposition, then there exists a smooth family of diffeomorphisms ψ_t such that $\psi^*\omega_t = \omega_0$. Most importantly, there exists open neighborhoods

N_0 and N_1 of Q and a diffeomorphism $\psi : N_0 \to N_1$ such that

$$\psi|_Q = id, \quad \psi^*\omega_1 = \omega_0$$

Now we just must show that τ is closed. Let T^*Q be normal bundle of Q in M, and let

$$E_\epsilon = \{(q, v) \in T^*Q| ||v|| \leq \epsilon\}$$

Recall that the exponential map

$$\exp : T^*Q \to M; \qquad \exp(q, v) = \frac{d}{dt}(\varphi_t(q))|_{t=0}$$

where φ_t is the local flow generated by v in a small neighborhood of q. By the proper choice of ϵ and shrinking $\mathcal{N}_0$ if necessary, exp is a diffeomorphism between $\mathcal{N}_0$ and E_ϵ. Hence we may define the map

$$\phi_t : \mathcal{N}_0 \to M; \qquad \phi_t(q, v) = \exp(q, tv)$$

which is a diffeomorphism for $t > 0$. Now $\phi_0(\mathcal{N}_0) \subset Q$ on which ω_0 and ω_1 agree, so

$$\phi_0^*\tau = \phi_0^*\omega_1 - \phi_0^*\omega_0 = 0$$

Clearly ϕ_1 is the identity, so ϕ_1^* is the identity, so

$$\phi_1^*\tau = \phi_1^*\omega_1 - \phi_1^*\omega_0 = \omega_1 - \omega_0 = \tau$$

Define the smooth family of vector fields

$$X_t = \left(\frac{d}{dt}\phi_t\right) \circ \phi_t^{-1}$$

Then, taking the derivative with to t yields

$$\frac{d}{dt}(\phi_t^*\tau) = \phi_t^*(\mathcal{L}_{X_t}\tau) = (d \circ i_{X_t} + i_{X_t} \circ d)\tau = d(i_{X_t}\tau)$$

So

$$\tau = \tau - 0 = \phi_1^*\tau - \phi_0^*\tau = \int_0^1 \frac{d}{dt}(\phi_t^*\tau)dt = \int_0^1 d(i_{X_t}\tau)dt = d\left(\int_0^1 (i_{X_t}\tau)dt\right)$$

Hence τ is exact and the desired result immediately follows.

Corollary 2.7. *(Darboux) Every symplectic form ω on M is locally diffeomorphic to the standard form ω_0 on $\mathbb{R}^{2n}$.*

To conclude, a classical mechanical system can be completely understood as a symplectic manifold with a chosen Hamiltonian function. Symplectic geometry is much more subtle than Riemannian geometry as symplectic manifolds have no local invarients. Finding and exploring global invarients is an active field to this day.

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