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Statement of the problem. Show that every sequence of $n^2 + 1$ distinct real numbers contains an increasing or decreasing subsequence of length $n + 1$.

Example. For $n = 2$, we have a sequence of length 5. If the sequence is 2, 3, 1, 5, 4; we have an increasing subsequence of length 3 (2, 3, 5).

Proof. (Pigeonhole Principle) Write a sequence as $(a_1, a_2, a_3, \dots, a_{n^2+1})$. For $k \in \{1, 2, 3, \dots, n^2 + 1\}$ let I_k be the length of the longest increasing subsequence which begins with a_k. Suppose that $I_k \leq n$ for all k; i.e. all increasing subsequences of the given sequence of length $n^2 + 1$ has at most length n. Notice that $I_k \geq 1$ for all k as well. By the pigeonhole principle, $n + 1$ of the numbers $I_1, I_2, I_3, \dots I_{n^2+1}$ must be equal. Thus we have $I_{k_1}, I_{k_2}, I_{k_3}, \dots, I_{k_{n+1}}$ such that $I_{k_1} = I_{k_2} = I_{k_3} = \dots = I_{k_{n+1}}$ where $k_1, k_2, k_3, \dots, k_{n+1} \in \{1, 2, 3, \dots, n^2 + 1\}$.

Now suppose that for some $m \in \{1, 2, 3, \dots, n\}$ we have $a_{k_m} < a_{k_{m+1}}$. Then we can take the longest increasing subsequence beginning with $a_{k_{m+1}}$ and put a_{k_m} in front to obtain a new increasing subsequence beginning with a_{k_m}. This implies that $I_{k_m} > I_{k_{m+1}}$, which contradicts our choices of I_{k_j}'s. Therefore, we must have $a_{k_m} > a_{k_{m+1}}$ for all $m \in \{1, 2, 3, \dots, n\}$. We conclude that

$$a_{k_2} > a_{k_3} > a_{k_3} > \dots > a_{k_{n+1}}.$$

This is a decreasing subsequence of length $n + 1$, which we wanted.

□

Proof. (Mathematical Induction) If $n = 1$, then we have a sequence of length 2. If this sequence has no increasing subsequence of length 2, then the entire sequence must be decreasing. Hence, the entire sequence forms a decreasing subsequence of length 2.

Now assume that the result holds for $n = k$. We must show that every sequence of $(k + 1)^2 + 1 = k^2 + 2k + 2$ distinct real numbers contain an increasing or decreasing subsequence of length $(k + 1) + 1 = k + 2$. Write a sequence as $(a_1, a_2, a_3, \dots, a_{k^2+2k+2})$ and let

$$\begin{aligned}
\mathrm{A} &= \{a_1, a_2, a_3, \dots, a_{k^2+2k+2}\},\\
\mathrm{B} &= \{a_j \in \mathrm{A} \mid a_j > a_i \text{ for all } a_i \in \mathrm{A},\ 0 < i < j\} \cup \{a_1\},\\
\mathrm{C} &= \{a_j \in \mathrm{A} \mid a_j < a_i \text{ for all } a_i \in \mathrm{A},\ 0 < i < j\} \cup \{a_1\}.
\end{aligned}$$

The terms in B and C clearly form an increasing and a decreasing sequence respectively. Therefore, we may assume that B and C both contain at most $k + 1$ terms. Now consider a set $\mathrm{A} - (\mathrm{B} \cup \mathrm{C})$. By our assumption, $\mathrm{B} \cup \mathrm{C}$ contains at most $2k + 2$ terms. Since B and C both contain a_1, $\mathrm{B} \cup \mathrm{C}$ contains at most $2k + 1$ terms. Thus, $\mathrm{A} - (\mathrm{B} \cup \mathrm{C})$ contains at least $k^2 + 1$ terms. By the inductive hypothesis, the terms contained in the set $\mathrm{A} - (\mathrm{B} \cup \mathrm{C})$ has an increasing or decreasing subsequence of length $k + 1$.

Notice that $a_1, a_2 \notin \mathrm{A} - (\mathrm{B} \cup \mathrm{C})$. Further notice that the terms $\mathrm{B} \cup \mathrm{C}$ are bounded by a_1 and a_2; i.e. WLOG assume $a_1 < a_2$, then

$$a_1 < a_i < a_2 \text{ for all } a_i \in \mathrm{B} \cup \mathrm{C}.$$

This means we can add a_1 or a_2 to the increasing or decreasing subsequence of length $k + 1$ respectively to form a new increasing or decreasing subsequence of length $k + 2$, which is what we wanted to show.

$\square$