The Graphs of Triangulations of Polygons

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Research Experience for Undergraduates Summer 2006

1 Basic Considerations

Let $\Gamma(n)$ be the graph with vertices being the labeled planar triangulation of a convex n-gon with the vertices only one the boundary. Let the edges be all the possible diagonal flips. A *diagonal flip* is defined for each internal edge, e, of J in $\Gamma(n)$. Since e is an internal edge, there are two faces, f_1 and f_2 that have e as an edge. Taken together, they form a quadrilateral with e as a diagonal. By flipping e to the other diagonal, e', and replacing f_1 and f_2 with the appropriate faces such that the boundary of the quadrilateral remains fixed, one creates a new triangulation J' in $\Gamma(n)$. This paper investigates the properties of the collection of graphs $\Gamma(n)$ for all n.

Definition 1.1. For a triangulation J in $\Gamma(n)$, the degree of a vertex v in J, $\deg(v)$, is the number of internal edges terminal at v.

Definition 1.2. An ear in a triangulation J in $\Gamma(n)$ is a face, f in J having a vertex v such that $\deg(v)$ is 0.

Definition 1.3. For some J in $\Gamma(n)$, each diagonal e partitions J. So J is said to be made of *subtriangulations* t_1 and t_2 glued along e. t_1 and t_2 each are triangulated, so they are homeomophic to something in $\Gamma(n_1)$ and $\Gamma(n_2)$. So by taking their representatives and topologically gluing them along e, they produce a triangulation homeomorphic to something in $\Gamma(n)$.

Lemma 1.4. *If J in $\Gamma(n)$ has two adjacent vertices each with degree 0 then $n = 3$.*

Proof. If J in $\Gamma(n)$ has two adjacent vertices v_i and v_{i+1} both with degree 0. Then there must be a single face, f that has $v_{i-1}, v_i, v_{i+1}, v_{i+2}$ as vertices. But f is a triangle, so, $v_{i-1} = v_{i+2}$ hence the only vertices are v_i, v_{i+1}, v_{i+2}. □

Lemma 1.5. *For any triangulation J in $\Gamma(n)$ with $n > 3$, J has at least 2 ears.*

Proof. We prove the lemma by induction. Let J be a triangulation in $\Gamma(4)$. Then clearly J is homeomorphic to two triangles glued along an edge. These two triangles are both ears. Inductively assume the lemma for all triangulations with less then n vertices. Let J be an element of $\Gamma(n)$ and let e be some diagonal of J. Then J can be considered the result of gluing j_1 and j_2 along e, where j_1 is everything to the left of e and j_2 is everything to the right of e. This is well defined since J is convex. Since e is an internal edge, j_1 and j_2 each have less then n vertices. By the inductive hypothesis, j_1 and j_2 each have two ears.

Now if both ends of e in j_1 are 0 degree vertices, then j_1 must be a triangle and therefore must have a third vertex also with degree equal to 0. Therefore gluing j_1 and j_2 effectively glues an ear onto j_2. If the ends of e in j_1 are both not ears then j_1 must have at least one ear away from the gluing region. Therefore when j_1 and j_2 are glued together, j_1 contributes at least one ear. Similarly j_2 contributes at least one ear, so J has at least 2 ears. □

Definition 1.6. A triangulation of an n-gon with all diagonals having one end terminal at a single vertex v is called *a triangulation in radial position at* v .

Theorem 1.7. $\Gamma(n)$ *is connected.*

Proof. Since each diagonal flip is reversible, it suffices to construct a path from any J in $\Gamma(n)$ to the radial position at v_1. We construct this sequence of flips by induction. The triangle is trivially in radial position. Let J be in $\Gamma(n)$, then either v_1 in J has degree 0 or not. If it does, then there is an edge e from v_n to v_2. Let flipping e be the edge τ in $\Gamma(n)$. If v_1 has a non zero degree, then either we are in radial position or not. If not, consider the internal edges $e_i, e_{i+1}, ..., e_{i'}$ that are terminal at v_1 as partitioning J, i.e. J is composed of the subtriangulations $j_i, j_{i+1}, ...j_{i'}$ glued along the partitioning edges. Since there is at least one diagonal e terminal at v_1, the number of vertices of f_i is strictly less then the then the number of vertices in J. By induction, there is a sequence of moves that transforms the triangulation of each subgraph into the subgraph in radial position at v_1. Since the diagonal flips in each partition are disjoint, they can be globally ordered to construct the desired sequence of diagonal flips. Thus each edge is terminal at v_1. □

Lemma 1.8. *The degree of each vertex in* $\Gamma(n)$ *is* $n-3$.

Proof. If e is an internal edge of some triangulation J in $\Gamma(n)$, then e is flippable. It suffices to show that there are $n-3$ internal edges of any normal, planar triangulation of polygon. We prove this by induction. The triangle trivially satisfies the lemma. Let J be in $\Gamma(n)$ for $n > 3$. Then since no two ears are adjacent, then J with an ear cut off, called J', is an element of $\Gamma(n-1)$. By induction J' has $n-4$ internal edges. However it is clear that by gluing the ear back on, that the edge that ear was cut off along becomes an internal edge thus J has $n-3$ internal edges. □

2 Dihedral Symmetry

In this section, it is shown that $\Gamma(n)$ quotiented by the action of the dihedral with $2n$ elements is ismorphic to the set $\Xi(n)$ of unlabeled triangulations. Then the size of $\Xi(n)$ is computed for each n.

Lemma 2.1. *The dihedral group with* $2n$ *elements act on* $\Gamma(n)$ *by relabeling the vertices of each triangulation* J *in* Γ.

Proof. By embedding each J in $\Gamma(n)$ in the polygon with vertices evenly spaced on the unit circle in $\mathbf{R}^2$ then letting d in D_{2n} act on $\mathbf{R}^2$ by a rotation and or a flip in the standard way $\Gamma(n)$ inherits the action of the ambient space. $\square$

Lemma 2.2. *The automorphism group of $\Gamma(n)$ is the dihedral group with $2n$ elements.*

Proof. An automorphism ϕ of $\Gamma(n)$ must send each J in $\Gamma(n)$ to some J' in $\Gamma(n)$. Let v_i be in J. Then let $w_i = \phi(v_i)$. Then since v_{i+1} is adjacent to v_i in J, w_{i+1} must be adjacent to w_i in J'. By extension, this specifies the image of all vertices and therefore the boundary of the polygon. Because a polygon is a disk, the automorphism can be extended to the whole triangulation. Since there are n choices for w_i with a cyclic relationship and 2 choices for w_{i+1}, this is the dihedral group. $\square$

Definition 2.3. The set of unlabeled triangulations with n vertices is $\Xi(n)$, the quotient of $\Gamma(n)$ under the dihedral action.

Definition 2.4. A self symmetry of a triangulation J is an element of a group that acts on a set of triangulations that that fix J.

Definition 2.5. Since $\Gamma(n)$ is isomorphic to the set of triangulations of the regular n-gon embedded in $\mathbf{R}^2$ with the vertices labeled clockwise with v_1 on the positive y-axis, the *center* is the the point $(0, 0)$.

Lemma 2.6. *The only self symmetries of a triangulation J in $\Gamma(n)$ are the self symmetries of the line or triangle containing the center.*

Proof. Since $Aut(\Gamma(n))$ is the dihedral group with $2n$ elements, every automorphism must fix the center. Since there are no internal vertices, the center must be contained in either a line or a triangle called c. Therefore for some J in $\Gamma(n)$, ϕ in $Stab(\Gamma(n))$, ϕ is in $Stab(center)$ so ϕ must send c to c, i.e. ϕ is in $Stab(c)$. $\square$

Lemma 2.7. *If J is in $\Gamma(n)$ with n even, then J does not have an axis of reflection that passes through an external edge away from a vertex.*

Proof. The proof follows by induction. The square with a diagonal cannot have an axis of symmetry that passes through an external edge away from a vertex, as the only axes of symmetry pass through the vertices. Assume that for all triangulations with an even number of vertices less than n, the lemma holds. Then let J be a triangulation with n vertices and an axis of reflection along a that passes through an external edge away from any vertex. The symmetry, ϕ across a must send vertices to vertices. Therefore there must be the same number of vertices on each side of a, and a does not intersect any vertex. If no internal edge crosses a, J contains a square, there must be an internal edge e that crosses a. Since e crosses a, $\phi(e) = e$. So e partitions J into t_1 and t_2, each closed under ϕ. Then, t_1 and t_2 each contain an even number of vertices as it otherwise breaks the symmetry. However, by induction, t_1 cannot have a as an axis of symmetry, so J does not have a as an axis of symmetry. $\square$

Theorem 2.8. *The number of element in $\Xi(n)$ is,*

$$|\Xi(n)| = \begin{cases} \frac{1}{2n}Ca(n) + \frac{1}{2}Ca(\frac{n+1}{2}) & 2 \not| n, 3 \not| n \\ \frac{1}{2n}Ca(n) + \frac{1}{2}Ca(\frac{n+1}{2}) + \frac{1}{3}Ca(\frac{n}{3}+1) & 3 \mid n, 2 \not| n \\ \frac{1}{2n}Ca(n) + \frac{3}{4}Ca(\frac{n}{2}+1) + \frac{1}{3}Ca(\frac{n}{3}+1) & 2 \mid n, 3 \mid n, 4 \not| n \\ \frac{1}{2n}Ca(n) + \frac{3}{4}Ca(\frac{n}{2}+1) & 2 \mid n, 3 \not| n \\ \frac{1}{2n}Ca(n) + \frac{3}{4}Ca(\frac{n}{2}+1) + \frac{1}{3}Ca(\frac{n}{3}+1) & 4 \mid n, 3 \mid n \end{cases}$$

where $Ca(n)$ is the Catalan number $\frac{\binom{2n}{n}}{n+1}$.

Proof. To compute $|\Xi(n)|$ it is sufficient to count the number of conjugacy classes of $\Gamma(n)$ under the dihedral action. Since the dihedral action fixes the center, and there are no vertices on the interior, the possible symmetries are limited by the symmetries of the cell containing the center.

Consider $\Xi(n)$ when neither 2 nor 3 divides n. Claim: no element in $\Xi(n)$ has any rotational symmetry. If the center of a triangulation k in $\Xi(n)$ is contained in a line e then any rotation must fix e. Therefore the only possible rotation is a rotation by 180 degrees, or reflection across an axis parallel or perpendicular to e. However these all induce a bijection on the vertices on either side of e, but since n is odd, this pairs sets of different parity, which is a contradiction. If the center is contained in a face f, then any rotation, or reflection. But by the transitivity of the action, all three of the subregions glued to the edges of f must have the same number of vertices, so the number of vertices must be divisible by 3, which contradicts the assumptions.

Let $\tilde{A}$ be all k in $\Xi(n)$ such that τ is a reflection across an axis of symmetry a. Then τ induces a bijection on the vertices on either side of a. If a contains 0 or 2 vertices, then since there is an odd number of vertices τ, pairs sets of different parity, which is a contradiction. Therefore a contains exactly 1 vertex v. Let f be the face containing a on it's interior and v, and let e_1 and e_2 be the edges of f terminal at v and e_3 the edge opposite v. Claim: e_3 is external. Assume that it is not, then let r_1, e_2 and r_3 be the subtriangulations glued along e_1, e_2 and e_3. Then r_2 must be the mirror image of r_2. Since r_1 and r_2 share v, the number of vertices in the union of r_1 and r_2 is odd. But since n is itself odd, r_3 must contain an even number of vertices. However, for τ to be a reflection of K, a must be a reflection of r_3. But a extends through e_3, a face of r_3 so a cannot be an axis of reflection for r_3, and e_3 is external. To count the size of $\tilde{A}$, observe that for each possible r_1 as considered an element of $\Gamma(\frac{n+1}{2})$, there is precisely 1 subtriangulation r_2 such that r_1 and r_2 glued appropriately to f is in $\tilde{A}$. No k in $\tilde{A}$ can have more then one axis of reflection. k certainly cannot have more then 3 axis of reflection as neither the line nor the triangle has more then 3. If k had 2 or 3 axes of reflection, then r_1, r_2 and r_3 must be equal, but e_3 is external, so f must be the triangle, but the triangle has 3 vertices and by assumption 3 does not divide n.

Let A be the set of all j in $\Gamma(n)$ congruent to some k in $\tilde{A}$. Then since each k in $\tilde{A}$ has exactly one axis of reflection $|A| = \frac{2n}{2}\tilde{A}$. If X is the set of all elements in $\Xi(n)$ without any symmetry, then

$$
\begin{aligned}
|\Xi(n)| &= \tilde{A} + X \\
&= \tilde{A} + \frac{|\Gamma(n)| - A}{2n} \\
&= |\Gamma(\frac{n+1}{2})| + \frac{|\Gamma(n)| - \frac{2n}{2}\tilde{A}}{2n} \\
&= \frac{1}{2n}Ca(n) + \frac{1}{2}Ca(\frac{n+1}{2}).
\end{aligned}
$$

Consider $\Xi(n)$ when 2 does not divide n but 3 does, then by the above case, for all $n > 3$, if k in $\Xi(n)$ has an axis of reflection then that is the only symmetry of k. So if $\tilde{A}$, and A are defined as above, then $\tilde{A}| = |\Gamma(\frac{n+1}{2})$ and $|A| = \frac{2n}{2}|\Gamma(\frac{n+1}{2})|$. Let $\tilde{B}$ be all b in $\Xi(n)$ that have rotation by 120 degrees as a symmetry. Then each b must consist of 3 copies of a subtriangulation t glued appropriately around a triangle containing the center. Observe that for each t taken from $\Gamma(\frac{n}{3}+1)$, it admits a triangulation, b, in $\tilde{B}$ (copy t appropriately). However since n is odd, $\tilde{A}$ and $\tilde{B}$ are disjoint for all $n > 3$, therefore the triangle b' obtained by reflecting b across an axis containing one vertex and the center of the opposite edge of the central triangle is conjugate but not equal to b. So $|\tilde{B}| = \frac{1}{2}|\Gamma(\frac{n}{3}+1)$. Since the orbit of rotation by 120 degrees is 3, B, the set of all j in $\Gamma(n)$ such that j is conjugate to something in $\tilde{B}$, has $\frac{2n}{3}\tilde{B}$ elements. If X is the set of all triangulations in $\Xi(n)$ without any symmetry, then

$$
\begin{aligned}
|\Xi(n)| &= \tilde{A} + \tilde{B} + X \\
&= \tilde{A} + \tilde{B} + \frac{|\Gamma(n)| - A - B}{2n} \\
&= |\Gamma(\frac{n+1}{2})| + \frac{1}{2}|\Gamma(\frac{n}{3}+1)| + \frac{|\Gamma(n) - \frac{2n}{2}|\Gamma(n+12)| - \frac{2n}{3}\frac{1}{2}|\Gamma(\frac{n}{3}+1)}{2n} \\
&= \frac{1}{2n}Ca(n) + \frac{1}{2}Ca(\frac{n+1}{2}) + \frac{1}{3}Ca(\frac{n}{3}+1).
\end{aligned}
$$

Consider $\Xi(n)$ when 2 and 3 divide n, while 4 does not. Let $\tilde{C}$ be all the k in $\Xi(n)$ that have 3 axes of reflection. Then k must consist of a central triangle with a triangle glued to each of its edges, and onto each of the 6 boundary edges is glued 6 copies of a triangulation s with $\frac{n}{6}+1$ vertices with every other one reflected across the axis perpendicular to gluing edge. k must be symmetric to itself under rotation by 120 degrees. Since this is all of the symmetries of the triangle, no action of the dihedral group on k can send k to itself without sending s to itself, there is a one-to-one correspondence between choices of s and elements in $\tilde{C}$ so $\tilde{C}$ has $Ca(\frac{n}{6}+1)$ elements. Since each k in $\tilde{C}$ represents 6 elements in $\Gamma(n)$, C, the set of elements in $\Gamma(n)$ conjugate to an element of $\tilde{C}$ has $\frac{2n}{6}Ca(\frac{n}{6}+1)$ elements.

Let B be the set of all k with rotation only by 120 and 240 degrees as its only symmetries. For some k to have these symmetries it must have three identical copies of a subtriangulation t with $\frac{n}{3}+1$ vertices glued appropriately around a central triangle. Some choices of t, however, admit a line of reflection perpendicular to each edge t that is glued to the central triangle. Since n has a factor of 2, $\frac{n}{3}$ has a factor of 2 so t has an odd number of vertices. Therefore from the considerations of the first case there are $\frac{\left(Ca(\frac{n}{3}+1)\right)+1}{2} = Ca(\frac{n}{6}+1)$ choices for triangulation of t that admit an axis of reflection. Since the rest of the triangulations do not admit an axis of reflection, one half of the remaining admit only rotation by 120 and 240 as their only symmetries. Therefore $|\tilde{B}| = \frac{1}{2}(Ca(\frac{n}{3}+1) - Ca(\frac{n}{6}+1))$. And since each k in $\tilde{B}$ represents 3 elements in $\Gamma(n)$, B, the set of elements in $\Gamma(n)$ congruent to something in $\tilde{B}$ has $\frac{2n}{3}\frac{1}{2}(Ca(\frac{n}{3}+1) - Ca(\frac{n}{6}{}_1))$ elements.

Let $\tilde{D}$ be the set all elements k in $\Xi(n)$, such that rotation by 180 degrees is a self symmetry. Then if for some k in $\tilde{D}$ the center was not contained on a diagonal then the face containing the center must be mapped to itself under rotation by 180 degrees, but a triangle does not have that symmetry. So, the center is contain on some edge e of k. If k posssessed an axis of reflection, then it must be either parallel to e or perpendicular to e and through the center. Since reflecting by one axis, rotating by 180 degrees and reflecting back across the axis is the same as reflecting across the axis perpendicular, k must have both axes of reflection. However, if k contains two perpendicular axes of reflection then k must be 4 copies of a subtriangle glued around a two triangles glued along the central triangle. Therefore the number of vertices of k must be divisible by 4, which by assumption it is not. Therefore, each element k in $\tilde{D}$ has only rotation by 180 degrees as a self symmetry. Let t_1 and t_2 be the 2 subtriangulations glued along e. Then for each choice of t taken from $\Gamma(\frac{n}{2}+1)$ there is exactly choice of t_2 such that k is in $\tilde{D}$. However reflecting k across the axis parallel to e or reflecting across the axis perpendicular to e produces a triangulation in the same orbit as k but not equal to k, and since e is fixed by both reflections, t_1 must be different as well. Therefore only half the choices of t_1 correspond with unique elements in $\tilde{D}$, that is, $|\tilde{D}| = \frac{1}{2}Ca(\frac{n}{2}+1)$. Since each element in $\tilde{D}$ represents 2 distinct elements in $\Gamma(n)$, D, the set of triangulations in $\Gamma(n)$ that are congruent to some element in $\tilde{D}$, has $\frac{2n}{2}\frac{1}{2}Ca(\frac{n}{2}+1)$ elements.

Let $\tilde{E}$ be the set of all elements k in $\Xi(n)$ that have a reflection across a central edge e as a self symmetry. Then for some k in $\tilde{E}$, k does not have any other symmetries as it would force n to be divisible by 4 but it is not by assumption. Therefore, if t_1 and t_2 are the subtriangulations glued along e, for each choice of t_1 taken from $\Gamma(n2+1)$ there is precisely 1 choice of t_2 such that k is in $\tilde{E}$. However rotating k by 180 degrees is not equal to k as it is not a symmetry, but is congruent to k as it is an element of the dihedral action since n is even. Also, since rotating by 180 degrees sends e to itself, t_1 must be different as well. Therefore only half of the choices of t_1 specify a unique element in $\tilde{E}$. So $|\tilde{E} = \frac{1}{2}Ca(\frac{n}{2}+1)$. Since each element in $\tilde{E}$ represents 2 distinct elements in $\Gamma(n)$, E, the set of triangulations in $\Gamma(n)$ that are congruent to some element in $\tilde{E}$, has $\frac{2n}{2}\frac{1}{2}CA(\frac{n}{2}+1)$ elements.

So far the triangulations with symmetry that have been counted include: triangulations with rotation by 120 and 240 degrees and 3 axes of symmetry; triangulations with only rotation by 120 and 240 degrees; triangulations with just rotation by 180 degrees; and triangulations with just reflection across a central line. Since the symmetries of a triangulation are limited by the symmetries of the cell containing the center, the cases that remain are: triangulations with a central line and one axis of symmetry perpendicular to the line and through the center; and triangulations with a central face with one axis of reflection extending through a vertex, the center and the center of the opposite edge. Both of these cases shall be dealt with at one time. Let $\tilde{F}$ be the set of all triangulations k in $\Xi(n)$ that have a reflection that fixes a line e perpendicular to axis a of reflection. Since n is even, a must intersect two distinct vertices or none at all. But if it intersects none, then by a lemma it cannot be an axis of symmetry.

Claim: A reflection can only fix at most one line perpendicular to a. Since a is a line of reflection all lines that cross a must be fixed. Say reflection across a fixes two edges e_1 and e_2 such that there are no edges in between e_1 and e_2 on a. Then they must either share both end points or non of the end points. If they share both end points then the region in between e_1 and e_2 forms a bigon and if they share non of the points then the region between e_1 and e_2 has more then three edges, neither of which are allowed in a triangulation.

Let F' be the set of labeled triangulations with an axis of symmetry a aligned vertically fixing a line e perpendicular to a. Claim: F' has $Ca(\frac{n}{2}+1)$ elements. To show the claim, a bijection between F' and the set of all labeled triangulations with an axis that fixes a perpendicular edge D' will be established. For some j in F', the endpoints of e and the end points of a form a quadrilateral with a and e along the diagonal. Flipping e aligns e with a. Since a and is an axis of symmetry, this produces the desired map. Observe that flipping e in different triangulations j_1 and j_2 cannot map to the same image in D' as nothing else in j_1 or j_2 was changed. Therefore the map is injective. To see that it is surjective, note that for each j in D' there must be two triangles t_1 and t_2 that share e as an edge. Then t_1 and t_2 are mapped to each other under reflection across e. Therefore the line connecting the vertices of t_1 and t_2 not on e, v_1 and v_2, must be perpendicular to e and fixed under reflection across e. So flipping e to (v_1, v_2) produces a triangulation with a single perpendicular edge fixed by the same reflection. In particular this shows that there are $Ca(\frac{n}{2}+1)$ such triangulations as there are $Ca(\frac{n}{2}+1)$ choices of triangulations t_1 that admit elements of D' and each one produces a unique labeled triangulation in D'.

Not every triangulation j in F' has no other symmetries. Since n is not divisible by 4, e cannot pass through the center so e cannot have an axis of reflection along it, however j may have two additional axes to be an element in $\tilde{C}$. For each j in $\tilde{C}$, orienting the central triangle with a vertex or an edge at the top gives two different elements of F' that are not in $\tilde{C}$. Since reflecting across the axis perpendicular to a and through the center and rotating by 180 degrees are both not symmetries, reflecting a triangulation j in F' across the axis perpendicular to a and through the center, or rotating by 180 degrees both give the same new triangulation j' in F'. So exactly half the elements in F' that are not in $\tilde{C}$ can be extended to unique triangulations in $\tilde{F}$. So $|\tilde{F}| = \frac{1}{2}(Ca(\frac{n}{2}+1) - 2Ca(\frac{n}{6}+1))$. Since each represents exactly 2 elements in $\Gamma(n)$, F, the set of element j in $\Gamma(n)$ that are congruent to element of $\tilde{F}$, has $\frac{1}{2}(\frac{1}{2}(Ca(\frac{n}{2}+1) - 2Ca(\frac{n}{6}+1)))$ elements.

Therefore together this shows,

$$\begin{aligned}
|\Xi(n)| &= \tilde{B} + \tilde{C} + \tilde{D} + \tilde{E} + \tilde{F} + \frac{Ca(n) - B - C - D - E - F}{2n} \\
&= \frac{1}{2}(Ca(\frac{n}{3}+1) - Ca(\frac{n}{6}+1)) + Ca(\frac{n}{6}+1) + \frac{1}{2}Ca(\frac{n}{2}+1) + \\
&\quad \frac{1}{2}Ca(\frac{n}{2}+1) + \frac{1}{2}(Ca(\frac{n}{2}+1) - 2Ca(\frac{n}{6}+1)) + \\
&\quad \frac{Ca(n) - \frac{2n}{3}\frac{1}{2}(Ca(\frac{n}{3}+1) - Ca(\frac{n}{6}+1)) - \frac{2n}{6}Ca(\frac{n}{6}+1) - \frac{2n}{2}\frac{1}{2}Ca(\frac{n}{2}+1) -}{} \\
&\quad \frac{\frac{2n}{2}\frac{1}{2}Ca(\frac{n}{2}+1) - \frac{1}{2}\frac{1}{2}(Ca(\frac{n}{2}+1) - 2Ca(\frac{n}{6}+1))}{2n} \\
&= \frac{1}{2n}Ca(n) + \frac{3}{4}Ca(\frac{n}{2}+1) + \frac{1}{3}Ca(\frac{n}{3}+1).
\end{aligned}$$

Consider $\Xi(n)$ when 2 divides n, while both 4 and 3 do not. For some k in $\Xi(n)$, if the center is contained in a face f, then the symmetries of k are a subset of the symmetries of f. Although f is symmetric under rotation by 120 and 240 degrees, the subtriangulations glued to f must all have the same number of vertices, but n is not divisible by 3 so k does not posses these symmetries. Also f has reflection across each line that extends from a vertex, through the center and through the center of the opposite face. But if k possessed any two of these, it would be symmetric under rotation by 120 degrees. So k has at most 1 one of these axes of reflection. If the center of k is contained in a line e, then the symmetries of k are some subset of the symmetries of e, rotation by 180 degrees, reflection across e, and reflection across the line perpendicular to e that passes through the center. If k has any 2 of these symmetries, it has all 3. In particular, if k has both reflections as symmetries, then k is made of a square with e a diagonal with 4 identical copies of a subtriangulation t glued appropriately around the boundary of the square. Then k has 4 times the number of vertices of t, but k is not divisible by 4. Therefore, k has at most one of the symmetries of the line.

Following the previous case, let $\tilde{D}$, D, $\tilde{E}$, E be defined similarly so $|\tilde{D}| = |\tilde{E}| = \frac{1}{2}Ca(\frac{n}{2}+1)$ and $D = E = \frac{2n}{4}Ca(\frac{n}{2}+1)$. $\tilde{F}$ can still be defined as the set of all k in $\Xi(n)$ that have an axis of reflection that fixes one edge perpendicular to the axis of reflection. Since, however, there are no triangulations with that have more then one axis of reflection, $|\tilde{F}| = \frac{1}{2}Ca(\frac{n}{2}+1)$ and let F be the set of all j in $\Gamma(n)$ that are conjugate to some k in $\tilde{F}$. So then $|F| = \frac{2n}{4}Ca(\frac{n}{2}+1)$.

Therefore for $\Xi(n)$ when 2 divides n, while both 4 and 3 do not,

$$
\begin{aligned}
|\Xi(n)| &= \tilde{D}+\tilde{E}+\tilde{F}+\frac{Ca(n)-D-E-F}{2n} \\
&= 3\frac{1}{2}Ca(\frac{n}{2}+1)+\frac{Ca(n)-3\frac{2n}{4}Ca(\frac{n}{2}+1)}{2n} \\
&= \frac{1}{2n}Ca(n)+\frac{3}{4}Ca(\frac{n}{2}+1).
\end{aligned}
$$

Consider $\Xi(n)$ when 4 divides n, while 3 do not. By the same argument used in the previous case, $\Xi(n)$ does not contain any elements 3 axes of symmetry or are symmetric under rotation by 120 and 240 degrees. Let $\tilde{G}$ be the set all triangulations in $\Xi(n)$ that have exactly 2 axes of reflection. Then as above, k in $\tilde{G}$ must have a central square with one axis of reflection along e the line containing the center and a diagonal of the square and the other axis the other diagonal perpendicular to e through the center. Then for each choice of a subtriangulation t glued on to the central square, there is precisely one choice for the remaining 3 subtriangulations such that the whole triangulation k is in $\tilde{G}$. But there are $Ca(\frac{n}{4}+1)$ choices of t. Since conjugation of one reflection by the other gives a rotation by 180 degrees, e has no symmetries that k does not, so each choice of t gives one of each element of $\tilde{G}$. Therefore $|\tilde{G}| = Ca(\frac{n}{4}+1)$, and G, the set of elements in $\Gamma(n)$ that are congruent to something in $\tilde{G}$, has $\frac{2n}{4}Ca(\frac{n}{4}+1)$ elements.

Following the previous case, let $\tilde{D}$, D, $\tilde{E}$, E, $\tilde{F}$, F be defined similarly, that is the sets of triangulations that have precisely one particular symmetry. To count the number of elements in $\tilde{D}$, consider the triangulations in $\Xi(n)$ with a central line e and 2 subtriangulations t_1 and t_2 glued on either side of e. Then for each t_1 taken from $\Gamma(\frac{n}{2}+1)$ there is exactly one way to choose t_2 such that the whole triangulation is symmetric when rotated by 180 degrees. However there are $Ca(\frac{n}{4}+1)$ choices of a subtriangulation s_1, one for each element in $\tilde{G}$, of t_1 such that t_1 is symmetric to itself when reflected across the line a perpendicular to e and through the center. The rest do not have a as an axis of symmetry, so reflecting each whole triangulation k across a is a triangulation different from k but conjugate to k. Therefore there $|\tilde{D}| = \frac{1}{2}(Ca(\frac{n}{2}+1) - Ca(\frac{n}{4}+1))$ and $|D| = \frac{2n}{4}(Ca(\frac{n}{2}+1) - Ca(\frac{n}{4}+1))$, by similar arguments to the ones above $|\tilde{E}| = |\tilde{F}| = |\tilde{D}|$ and $|E| = |F| = |D|$.

Therefore when 4 divides n, while 3 do not, $\Xi(n)$ has that,

$$
\begin{aligned}
|\Xi(n)| &= \tilde{D}+\tilde{E}+\tilde{F}+\tilde{G}+\frac{Ca(n)-D-E-F+G}{2n} \\
&= 3\frac{1}{2}(Ca(\frac{n}{2}+1)-Ca(\frac{n}{4}+1))+Ca(\frac{n}{4}+1)+ \\
&\quad \frac{Ca(n)-3\frac{2n}{4}(Ca(\frac{n}{2}+1)-Ca(\frac{n}{4}+1))-\frac{2n}{4}Ca(\frac{n}{4}+1)}{2n} \\
&= \frac{1}{2n}Ca(n)+\frac{3}{4}Ca(\frac{n}{2}+1).
\end{aligned}
$$

Since this is the same result as when simply 2 divides n while neither 4 nor 3 do, it will be subsumed under that case in the result.

Consider $Xi(n)$ when 4 divides n and 3 divides n. Then combining the work of the previous cases gives,

$$\begin{aligned}
|\Xi(n)| &= \tilde{B}+\tilde{D}+\tilde{E}+\tilde{F}+\tilde{G}+\frac{Ca(n)-B-D-E-F-G}{2n} \\
&= \frac{1}{2}(Ca(\frac{n}{3}+1)-Ca(\frac{n}{6}+1))+Ca(\frac{n}{6}+1)+\frac{1}{2}(Ca(\frac{n}{2}+1)- \\
&\quad Ca(\frac{n}{4}+1))+\frac{1}{2}(Ca(\frac{n}{2}+1)-Ca(\frac{n}{4}+1))+ \\
&\quad \frac{1}{2}(Ca(\frac{n}{2}+1)-2Ca(\frac{n}{6}+1)-Ca(\frac{n}{4}+1))+Ca(\frac{n}{4}+1) \\
&\quad \underline{Ca(n)-\tfrac{2n}{3}\tfrac{1}{2}(Ca(\tfrac{n}{3}+1)-Ca(\tfrac{n}{6}+1))-\tfrac{2n}{2}\tfrac{1}{2}(Ca(\tfrac{n}{2}+1)} \\
&\quad \underline{-Ca(\tfrac{n}{4}+1))-\tfrac{2n}{2}\tfrac{1}{2}(Ca(\tfrac{n}{2}+1)-Ca(\tfrac{n}{4}+1))-\tfrac{1}{2}\tfrac{1}{2}(Ca(\tfrac{n}{2}+1)-} \\
&\quad \frac{2Ca(\frac{n}{6}+1)-Ca(\frac{n}{4}+1))-\frac{2n}{4}Ca(\frac{n}{4}+1)-\frac{2n}{6}Ca(\frac{n}{6}+1)}{2n} \\
&= \frac{1}{2n}Ca(n)+\frac{3}{4}Ca(\frac{n}{2}+1)+\frac{1}{3}Ca(\frac{n}{3}+1),
\end{aligned}$$

which completes the proof of the theorem. $\square$