NEWTON'S METHOD

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1. Abstract

Newton's Method is one of the most powerful tools available for finding the roots of polynomials. Given an initial guess for the root, the Newton Iteration Function, provided that initial properties are satisfied, will often converge to a root. This paper will address connections between the roots of a polynomial to the fixed points of the Newton Iteration Function. In the cases where the Iteration Function converges to a root, the speed of convergence will also be determined. It will be shown that a root with multiplicity 1 converges quadratically, while a root with multiplicity greater than 1 converges linearly.

2. Newton's Method Is An Iterated Function

Given a differentiable function $F(x), x \in \mathbb{R}$ it is possible to find where $F(x) = 0$ by making an initial guess and then applying Newton's Method. First make a guess for the root and call it x_0. Find an equation to the tangent line at $(x_0, F(x_0))$. Since the slope is $F'(x_0)$ the equation of the tangent line is of the form

$$\begin{aligned} y &= F'(x_0)x + B \\ B &= F(x_0) - F'(x_0)x_0 \\ \Rightarrow y &= F'(x_0)x + F(x_0) - F'(x_0)x_0 \\ y &= F'(x_0)(x - x_0) + F(x_0) \end{aligned}$$

Next, set $y = 0$ and find the root of this tangent line, which will be denoted by x_1.

$$\begin{aligned} 0 &= F'(x_0)(x_1 - x_0) + F(x_0) \\ x_1 &= x_0 + \frac{F(x_0)}{F'(x_0)} \end{aligned}$$

This process can be continued, until a root is reached, with each process being referred to as an *iteration*

Definition 1 (Newton Iteration Function)**.** *The Newton Iteration Function is defined by*

$$N(x) = x - \frac{F(x)}{F'(x)}$$

Frequently, but not always, the orbit of a point x_0 converges to a root of the polynomial.

Definition 2 (Orbit)**.** *The orbit of an iterated function is defined as the sequence* $\{f^{on}(x)\}_{n\to\infty}$*, where* f^{on} *denotes the nth iteration of* f *starting at* x*.*

In the case of Newton's Method the initial guess of the root of the polynomial is the point whose orbit we focus on.

3. The Fixed Point Theorem And The Failure of Newton's Method

Depending on the function under consideration and the initial point selected, the orbit of this point may not always converge. For example, in order for Newton's Method to be successful it is required that for all x such that $F(x) = 0$, $F'(x) \neq 0$, as this leaves the Newton Iteration Function undefined. In addition, this initial guess must be sufficiently close to the root.

According to the Newton-Raphson Fixed Point Theorem the roots of a function, F are also fixed points of the Newton Iteration function, and vice versa

Definition 3 (Fixed Point)**.** *A fixed point x is a point such that given a function G, $G(x) = x$.*

Definition 4 (Multiplicity)**.** *A root x_0 has multiplicity k iff $f^{[k-1]}(x_0) = 0$ and $f^k(x_0) \neq 0$, where $f^{[k]}$ denotes the kth derivative of f at x_0.*

Definition 5 (Attracting Point)**.** *x_0 is an attracting point of a function F if it satisfies the inequality*

$$|F'(x_0)| < 1.$$

Definition 6 (Repelling Point)**.** *A repelling point x_0 satisfies the inequality*

$$|F'(x_0)| > 1.$$

Lemma 1. *If x_0 is a root of multiplicity k of F, then $F(x)$ can be written as*

$$F(x) = (x - x_0)^k G(x)$$

Where G is a continuous function such that $G(x_0) \neq 0$.

Theorem 1. *The Fixed Point Theorem states that if F is a function with Newton Iteration Function N then x_0 is a root of F with multiplicity k if and only if x_0 is an attracting fixed point of N.*

Proof. For the case when a root x_0 of a function $F(x)$ has multiplicity 1, then by definition $F'(x_0) \neq 0$ and $F(x_0) = 0$. From Newton's Iteration Formula it follows that

$$f(x_0) = 0 \Rightarrow N(x_0) = x_0$$

So x_0 is a fixed point of N. Conversely, if x_0 is a fixed point of N then $N(x_0) = x_0$ which implies that $F(x_0) = 0$ proving that x_0 is a root of F To prove the more general case where a root x_0 has multiplicity $k > 1$ apply Lemma 1 Suppose a function $F(x)$ has a root x_0 of multiplicity $k > 1$ then $F(x) = (x - x_0)^k G(x)$ Taking the derivative,

$$F'(x) = k(x - x_0)^{k-1} G(x) + (x - x_0)^k G'(x)$$

Newton's Iteration Function then becomes

$$N(x) = x - \frac{(x - x_0)^k G(x)}{k(x - x_0)^{k-1} G(x) + (x - x_0)^k G'(x)} \tag{1}$$

$$N(x) = x - \frac{(x - x_0) G(x)}{kG(x) + (x - x_0) G'(x)} \tag{2}$$

Substituting the root x_0 into (2),

$$N(x_0) = x_0$$

proving that a root of multiplicity $k > 1$ is a fixed point of $N(x)$ To show that it is an attracting point, we take the derivative of $N(x)$

$$N'(x) = \frac{k(k-1)(G(x))^2 + 2k(x-x_0)G(x)G'(x) + (x-x_0)^2G(x)G''(x)}{k^2(G(x))^2 + 2k(x-x_0)G(x)G'(x) + (x-x_0)^2(G'(x))^2}$$

Applying $x = x_0$ gives

$$N'(x_0) = \frac{k(k-1)(G(x))^2}{k^2(G(x))^2}$$

$$\Rightarrow N'(x_0) = \frac{k-1}{k} < 1$$

So x_0 is an attracting fixed point of N. Conversely, if a point x is a fixed point of N then it is a root of F. By equation (2) the numerator

$$(x-x_0)G(x) = 0 \Rightarrow x = x_0$$

Because, by definition, $G(x) \neq 0$. [1] □

4. The Speed of Convergence of Newton's Method

Definition 7 (Linear Convergence)**.** *If a sequence $\{x_k\}$ converges to a root R then we say it converges linearly to R if*

$$\lim_{k\to\infty} \frac{|x_{k+1} - R|}{|x_k - R|} = \mu.$$

where $0 < \mu < 1$

Definition 8 (Quadratic Convergence)**.** *If a sequence converges quadratically to a root R then*

$$\lim_{k\to\infty} \frac{|x_{k+1} - R|}{|x_k - R|^2} = \mu.$$

where $\mu > 0$

Lemma 2. *If a root x_0 has multiplicity $k > 1$ and the orbit of Newton's Iteration Function converges, then the orbit converges linearly.*

Lemma 3. *If a root x_0 has multiplicity 1 and the orbit of Newton's Iteration Function converges, then the orbit converges quadratically.*

Proof. To prove that Newton's Method converges quadratically for a root of mulitiplicity 1, we first express

$$\lim_{k\to\infty} \frac{|x_{k+1} - R|}{|x_k - R|^2}$$

as

$$\lim_{k\to\infty} \frac{|E_{k+1}|}{|E_k|^2}$$

where E represents the error term. Consider the Taylor Polynomial of a function $f(x)$ whose roots we wish to compute around the point x_k. Assume that $f'(x_k) \neq 0$ then

$$f(x) = f(x_k) + f'(x_k)(x - x_k) + \frac{f''(x_k)(x-x_k)^2}{2} \tag{3}$$

If $f(x)$ has a root at $x = x_r$ then if x_r is substituted for x then (3) becomes

$$\begin{aligned} 0 &= f(x_k) + f'(x_k)(x_r - x_k) + \frac{f''(x_k)(x_r - x_k)^2}{2} \\ \Rightarrow \frac{-f''(x_k)(x_r - x_k)^2}{2} &= f(x_k) + f'(x_k)(x_r - x_k) \end{aligned}$$

Dividing by $f'(x_k)$

$$\frac{-f''(x_k)(x_r - x_k)^2}{2} = \frac{f(x_k)}{f'(x_k)} + (x_r - x_k)$$

Applying the iteration formula, $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$ gives

$$\begin{aligned} \frac{-f''(x_k)(x_r - x_k)^2}{2} &= x_r - x_{k+1} \\ \Rightarrow \frac{-f''(x_k)(E_k)^2}{2} &= E_{k+1} \end{aligned}$$

If, in fact, the orbit does converge to a root then $\lim_{k\to\infty} f'(x_k) = f'(x_r)$ and $\lim_{k\to\infty} f''(x_k) = f''(x_r)$ So that

$$\begin{aligned} \frac{E_{k+1}}{E_k^2} &= \frac{-f''(x_k)}{2f'(x_k)} \\ \Rightarrow \lim_{k\to\infty} \frac{|E_{k+1}|}{|E_k|^2} &= \frac{|-f''(x_r)|}{|2f'(x_r)|} > 0 \end{aligned}$$

Note that $f'(x_r) \neq 0$ because x_r has mulitiplicity 1. If x_r has mulitiplicity $k > 1$ then, applying Lemma 1, $f(x)$ can be written as $f(x) = (x - x_r)^k G(x)$. Therefore,

$$f'(x) = k(x - x_r)^{k-1} G(x) + (x - x_r)^k G'(x) \Rightarrow f'(x_r) = 0.$$

[2]

□

References

[1] Devaney, Robert. *A First Course In Chaotic Dynamical Systems.* Boston:Addison-Wesley, 1992.

[2] Mathews, John. "The Accelerated and Modified Newton Methods."2003. Fullerton University. 24 July. 2006.