A Walking Tour of Microlocal Analysis

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Abstract

We summarize some of the basic principles of microlocal analysis and their applications. After reviewing distributions, we then define pseudodifferential operators, their symbols, and the pseudolocal property. This then leads to the fundamental notion of microlocal analysis: the wave front set of a distribution. The wave front set will then be used to analyze the problem of the propagation of singularities.

1 Introduction

The techniques of microlocal analysis were developed in the 1960's and 70's as part of the study of linear partial differential equations. Many of the ideas are due to the work of Hormander, Kohn and Nirenberg, and Maslov, in which they generalized existing notions from analysis to investigate distributions and their singularities. Indeed, much of microlocal analysis is concerned with shifting the study of a distribution's singularities from the base space to the cotangent bundle. Such a study will allow us to answer basic questions such as when the product of two distributions is well-defined, as well as extending and strengthening some standard theorems from differential equations, such as elliptic regularity.

2 Review of Distributions

In elementary calculus, one is immediately confronted with functions that are not differentiable. The introduction of distributions seeks to remedy this shortcoming by providing the smallest set of objects such that every member is infinitely differentiable in a sense to be made precise later (hence the term "generalized function"). In this section, we will give a precise definition and topology of this space of functions. Fix an open set $\Omega \subset \mathbf{R}^n$, and consider the space of smooth functions with compact support $C_c^\infty(\Omega)$. We would like to give this space a suitable topology under which it will be complete. This can be accomplished by introducing a family of seminorms that range over all compact $K \subset \Omega$:

$$P_{K,m} = \sup_{|\alpha| \leq m, x \in K} |D^\alpha f(x)|, \tag{1}$$

where

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} \partial^{\alpha_2} \ldots \partial^{\alpha_k}}; \quad \|\alpha\|_{l_1} = |\alpha|, \text{and } \operatorname{supp}(f) \subseteq K.$$

Because this family of seminorms $\{P^j\}$ separates points in that $P^j(x)$ for all j implies $x = 0$, they turn $C_c^\infty(\Omega)$ into a locally convex space, such that the natural metrizable topology is Hausdorff. It is well-known that $C_c^\infty(\mathbf{R}^n)$ is not complete under this topology, but it can be made so by equipping it with the "strict inductive limit topology"[1]. To get the largest space of such functions contained in $C_C^\infty(\Omega)$, we take the union of all sets of smooth functions whose support is contained in a compact subset of Ω. We define this space to be $D(\Omega) = \bigcup_{K \subset \Omega} C_c^\infty(K)$, where K is compact.

We have now made the space of "test functions" $D(\Omega)$ into a Fréchet space, which gives us an acceptable notion of what it means for a sequence to converge in this space, i.e. a notion of continuity. More precisely, a sequence $\{\phi_j\} \in D(\Omega)$ converges to $\phi \in D(\Omega)$ if $\operatorname{supp}(\phi_j)$ is contained in some compact $K \subset \Omega$ for all j, and $P_{m,k}(\phi_j - \phi) \to 0$. A distribution is then defined to be a continuous linear functional on this space of test functions. The space of all distributions is denoted $D'(\Omega)$. In other words, a linear functional T on $D(\Omega)$ is a distribution if and only for every compact $K \subset \Omega$, there is a constant $C > 0$ and $n \in \mathbf{N}$ such that

$$T(\varphi) \leq C \sup_{|\alpha| \leq n, x \in K} |D^\alpha \varphi(x)|, \text{for } \varphi \in D(\Omega). \tag{2}$$

Even though we have defined distributions on open subsets of $\mathbf{R}^n$, they can easily be extended to a smooth manifold M. The construction proceeds in the obvious way, by defining distributions on the homeomorphic images of coordinate patches. More specifically, if (U_j, ϕ_j) is an arbitrary chart, then consider the distribution $u_j \in$

[1]See p. 146 of [4] for a discussion of strict inductive limits

$D'(\phi_j(U_j))$. A distribution on M is given by all such local representatives that satisfy $u_j \circ (\phi_j \circ \phi_i^{-1}) = u_i$ on $\phi_i(U_i \cap U_j)$. It is clear that this definition coincides with our previous definition in the case that M is an open subset of $\mathbf{R}^n$. Furthermore, it can be shown that this definition is independent of the charts we choose.

Finally, we define the notion of a "weak" derivative, a construct that allows every distribution to be infinitely differentiable. The basic idea in this definition is that since the test functions are smooth and vanish outside a suitably large space, then we can integrate by parts and discard the boundary terms. More formally, the weak derivative of $T \in D'(\Omega)$ is defined as the distribution $D^\alpha T$ that satisfies

$$\int (D^\alpha T)(x)\varphi(x)dx = (-1)^{|\alpha|} \int T(x)(D^\alpha \varphi)(x)dx \quad \forall \varphi \in D(\Omega). \tag{3}$$

3 Pseudodifferential Operators and Symbols

In this section we define some of the basic properties of pseudodifferential operators and the symbols associated with them. To motivate the definition of symbols, consider the following result from ordinary real analysis. Let $p(x, D) = \sum_{|\alpha| \le k} a_\alpha(x) D_x^\alpha$ be a differential operator, where the coefficients $a_\alpha(x)$ are smooth functions, $D_x^\alpha = D_{x_1}^{\alpha_1} \dots D_{x_n}^{\alpha_n}$, and $D_{x_j} = -i\partial_{x_j}$. If f is a function of rapid decrease, then the Fourier inversion formula is valid, and we can write f as an integral of its $\hat{f}$. Now apply $p(x, D)$ to f to get:

$$p(x, D)f(x) = \frac{1}{(2\pi)^n} \int p(x, \xi) e^{-ix\cdot\xi} \hat{f}(\xi) d\xi. \tag{4}$$

We will see that (4) eventually will define a pseudodifferential operator as a map from $C_c^\infty(X)$ to C^∞. To make this definition more precise, however, we first need to specify what $p(x, \xi)$ is. Since we have applied a differential operator that is a polynomial in derivatives, (4) suggests that $p(x, \xi)$ (called the characteristic polynomial of the differential operator) is not just an arbitrary function, but a polynomial in some vector. Since $p(x, D)$ takes derivatives with respect to x, every time we apply $p(x, D)$ to $f(x)$, we pick up more powers of ξ; note that there are no factors of i present due to our definition of D_x^α. In particular, $p(x, \xi)$ can be replaced by $\sum_{|\alpha| \le k} a_\alpha(x)\xi^\alpha$, which is now a type of function instead of a differential operator. We would like this $p(x, \xi)$ to satisfy a certain bound such that every time we differentiate it, we lose a degree of smoothness. This leads to the definition of symbols, a class of functions that plays an important role in microlocal analysis.

Definition 3.1 *Let $\Omega \subset \boldsymbol{R}^q$ be open, $\delta, \eta \in [0,1]$, $m \in \boldsymbol{R}$, and n a positive integer. Then the space of symbols of order m and type (η, δ), denoted $S^m_{\eta,\delta}(\Omega \times \boldsymbol{R}^n)$, is the set of all $u \in C^\infty(\Omega \times \boldsymbol{R}^n)$ such that for all compact $K \subset \boldsymbol{R}^n$ and all $\alpha \in \boldsymbol{N}^q$ and $\beta \in \boldsymbol{N}^n$, there is a constant $C = C(K, \alpha, \beta)$ such that*

$$|\partial_x^\alpha \partial_\theta^\beta u(x,\theta)| \leq C(1+|\theta|)^{m-\eta|\beta|+\delta|\alpha|}. \tag{5}$$

We denote the intersection of $S^m_{\eta,\delta}$ for all real m by $S^{-\infty}_{\eta,\delta}$.

Definition 3.2 *Let $p(x,\xi) \in S^m_{\eta,\delta}$. Then a pseudodifferential operator A is a function from $C_c^\infty(X)$ to $D'(X)$ defined by (4). The space of all such operators is denoted $L^m_{\eta,\delta}$.*

$S^m_{\eta,\delta}(\Omega \times \mathbf{R}^n)$ can be made into a Fréchet space by introducing the seminorms:

$$P_{K,\alpha,\beta}(u) = \sup_{(x,y)\in K\times\mathbf{R}^n} \frac{|\partial_x^\alpha \partial_y^\beta u(x,y)|}{(1+|y|)^{m-\eta|\beta|+\delta|\alpha|}}.$$

According to our definition, pseudodifferential operators are only functions on $C_c^\infty(X)$; we do not know if they can be defined on more general spaces, e.g. the space of distributions. In order to determine when such an extension is allowed, we must consider the "kernel" of the operator. In the above expression for pseudodifferential operators, expand the function $\hat{f}(\xi)$ in terms of $f(\xi)$ using the definition of the transform. This suggests that the kernel of the pseudodifferential operator A, denoted K_A, is given by an "oscillatory integral" of the form

$$\frac{1}{(2\pi)^n}\int p(x,\xi)e^{i(x-y)\cdot\xi},$$

where $p(x,\xi)$ is a symbol of order m. This is indeed the case, and it can be made more rigorous using the Schwartz kernel theorem[2], which establishes a one-one correspondence between distributions on $X \times Y$ and linear maps $C_c^\infty(Y) \to D'(X)$. The correspondence that defines the kernel can be written as

$$< Au, v >=< K_A, u(y)v(x) >= \frac{1}{(2\pi)^n}\iiint u(y)p(x,\xi)v(x)e^{i(y-x)\cdot\xi}dydxd\xi. \tag{6}$$

This leads to the following important lemma:

[2] For a statement and proof of the Schwartz kernel theorem, see §5.2 of [3].

Lemma 3.3 *If $\eta > 0$, then K_A is a smooth function off the diagonal in $\boldsymbol{R}^n \times \boldsymbol{R}^n$.*

Proof : For an arbitrary $\beta > 0$, we can integrate by parts β-times the integral for K_A and discard the boundary terms to get

$$(y-x)^\beta K_A = \int e^{i(y-x)\cdot\xi} D_\xi^\beta p(x,\xi)d\xi. \tag{7}$$

This integral converges when β is chosen large enough so that $m + n - \eta|\beta| < 0$. If we differentiate the above equation q-times, where $m + q + n - \eta|\beta| < 0$, then the integral also converges, implying that $(y-x)^\beta K_A \in C^q(\mathbf{R}^n \times \mathbf{R}^n)$.□

This lemma implies that if $K_A \in C^\infty(X \times Y)$, then A can be extended to a continuous map from $D'_c(Y)$ to $C^\infty(X)$ (D'_c denotes the space of compactly supported distributions). Before stating an important corollary of this lemma, we need some definitions that will allow us to characterize singularities of a distribution:

Definition 3.4 *If $u \in D'(X)$, then the singular support of u, denoted sing supp u, is the smallest closed subset of X on which u is not C^∞.*

Definition 3.5 *$A \in L^m_{\eta,\delta}(\Omega)$ is properly supported if supp $K_A = C \subset X \times Y$ is proper, i.e. if the projections $\pi_x : C \to X$ and $\pi_y : C \to Y$ are proper maps.*

It is clear that if $A \in L^m_{\eta,\delta}(\Omega)$ is properly supported, then it is a continuous map from $D'(\Omega)$ to $D'(\Omega)$.

Corollary 3.6 *Let $A \in L^m_{\eta,\delta}$ and u be a compactly supported distribution on Ω. If $\eta > 0$, then sing supp $Au \subset$ sing supp u.*

Proof : Let $A \in L^m_{\eta,\delta}(\Omega)$ and $u \in D'_c(\Omega)$. Consider some open set U in sing supp u and a test function ϕ=1 on sing supp u. We can "localize" u by multiplying it by the test function, which gives two new distributions $u_1 = \phi u$ and $u_2 = (1-\phi)u$ so that $u = u_1 + u_2$. Let K_A be the kernel associated with A. By the lemma above, $K_A(x,y)$ is C^∞ off the diagonal, i.e. when $x \notin V$ and $y \in V$. Since Au is just K_A smeared with u, we have

$$Au_1 = \int_{supp\ u_1} K_A(x,y)u_1(y)dy. \tag{8}$$

We know that the kernel is smooth, so apply the operator D^α to both sides to get

$$D^\alpha Au_1 = \int_{supp\ u_1} D^\alpha K_A(x,y)u_1(y)dy. \tag{9}$$

This implies that $D^\alpha A$ is smooth off the diagonal as well, hence it is smooth outside of V. Since u_2 is C_c^∞ on Ω, it follows that $Au_2 \in C^\infty(\Omega)$. Because pseudodifferential operators are linear maps, $Au_1 + Au_2 = A(u_1 + u_2) = Au$, implying that Au is smooth outside of V. But since we chose V to be an arbitary open subset of sing supp u, this implies that Au is smooth outside of sing supp u.□

This corollary is often called the *pseudolocal property* of operators, and it will play an important role in the discussion of wave front sets below.

Now we give a generalized notion of an elliptic differential operator in the context of pseudodifferential operators. Recall that if $p(x, D)$ is a differential operator with smooth coefficients

$$p(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D_x^\alpha, \tag{10}$$

then the *principal part* of $p(D)$ is defined to be

$$P_m(x, D) = \sum_{|\alpha| = m} a_\alpha(x) D_x^\alpha. \tag{11}$$

Similarly, if $p(x, \xi) \in S_{\eta,\delta}^m(\Omega)$ as above, then we define the *principal symbol* to be the symbol

$$p_m(x, \xi) = \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha. \tag{12}$$

The operator $p(x, D)$ is said to be elliptic if $p_m(x, \xi) \neq 0$ for all $0 \neq \xi \in \mathbf{R}^n$. Pseudodifferential operators generalize this idea by extending the same notion to the associated symbol. An operator $p(x, D) \in L_{\eta,\delta}^m$, is said to be elliptic at the point (x_0, ξ_0) if there is some conical neighborhood V of (x_0, ξ_0) and a positive constant C such that

$$|p(x, \xi)| \geq C(1 + |\xi|)^m \tag{13}$$

for all $(x, \xi) \in V$ and $|\xi| \geq C$.

If we restrict our attention to elliptic operators, we can make the estimate of the pseudolocal property more precise by showing that the singular supports are equal. Before proving this theorem, we first need a definition:

Definition 3.7 *If P is a pseudodifferential operator of order m, then a parametrix for P is a properly supported pseudodifferential operator $Q \in L_{\eta,\delta}^{-\infty}$ such that $PQ - I \in L_{\eta,\delta}^{-\infty}$ and $QP - I \in L_{\eta,\delta}^{-\infty}$. It is true that if P is elliptic, then P has a parametrix $Q \in L_{\eta,\delta}^{-m}$ (See [1] p. 298 for details).*

Theorem 3.8 *If $P \in L^m_{\eta,\delta}(\Omega)$ is elliptic and u is a distribution on Ω, then sing supp (Pu)=sing supp (u).*

Proof : We already know that sing supp $(Pu) \subset$ sing supp (u), which amounts to saying that if $u \in D'(\Omega)$ is smooth on some open $V \subset \Omega$, then Pu is smooth on V. To show the converse inclusion, we need to show that if $Pu \in C^\infty(V)$, then $u|_V$ is smooth. To show this, let Q be a properly supported parametrix for P. Then the restriction of its kernel K_Q to $V \times V$ is also properly supported. Because $Pu|_V \in C^\infty(V)$, it follows that $QPU|_V \in C^\infty(V)$ and K_Q is smoothing (i.e. $K_Q u \in C^\infty$). As a result, $u|_V \in C^\infty(V)$. So we have shown that sing supp $(u) \subset$ sing supp (Pu) for the case of an elliptic operator. Combining this with Corollary 3.6 shows that equality must hold. □

Note that a key result used in the proof was that every elliptic operator has a parametrix and has a kernel that is smooth off the diagonal.

4 Microlocal Analysis

It is well known that the decay properties of the Fourier transform of a distribution are related to its smoothness. More specifically, if $u \in D'(\Omega)$, then we can localize u by multiplying it by a test function φ and considering φu. By a basic theorem of Fourier analysis and the Paley-Wiener theorem, $\widehat{\varphi u}$ is an entire holomorphic function, and it is smooth if it is of rapid decrease. The idea of the wave front set is to characterize both the points *and* the directions in which this condition fails to hold. Even though this construct is most easily motivated by considering the Fourier transforms of localized distributions, we find it more satisfying to use our above ideas of pseudodifferential operators. Instead, our definition of the smoothness of u will be the following: (In this section, all pseudodifferential operators are assumed to be properly supported)

Definition 4.1 *If $u \in D'(\Omega)$ and $(x,\xi) \in T^*\Omega$, then u is smooth on a neighborhood of (x,ξ) if there is some $A \in L^m_{\eta,\delta}(\Omega)$ that is elliptic at (x,ξ), and such that $Au \in C^\infty(\Omega)$.*

This suggests that the set of points where u is not smooth is related to the set of points where some pseudodifferential operator A is not elliptic. Indeed, this is the case. We now define the *characteristic variety* of $A \in L^m_{\eta,\delta}(\Omega)$ to be

$$\mathrm{Char}\, A = \{(x,\xi) \in T^*\Omega \,|\, \text{A is not elliptic at } (x,\xi)\}.$$

From here, it is easy to define the wave front set of a distribution, a concept that plays a central role in microlocal analysis.

Definition 4.2 *If $u \in D'(\Omega)$, then the wave front set of u, denoted WF(u), is given by $\bigcap \{ Char\, A \,\Big|\, A \in L^m_{1,0}, \text{ and } Au \in C^\infty \}$.*

It is immediate that the wave front set is a refinement of the notion of singular support considered above, as it also describes singular directions. Moreover, the wave front set also has the advantage of being a closed conic subset of the cotangent bundle. Because of this, it is clear that we can recover the singular support of a distribution by merely projecting the wave front set onto its first factor:

Lemma 4.3 *Let $\Pi : T^*X \to X$ be the projection map defined by $\Pi(x, \xi) = x$ for all $(x, \xi) \in T^*X$. Then $\Pi(WF(u))$ =sing supp u.*

Using the following definition, we also have a microlocal analogue of the pseudolocal property from above;

Definition 4.4 *For $P \in L^m_{\eta,\delta}$, define the microsupport of P, denoted μsupp(P), to be the complement of the set on which P is smoothing.*

Theorem 4.5 *If $P \in L^m_{\eta,\delta}(\Omega)$, $u \in D'(\Omega)$, then WF(Pu) $\subset$ WF(u)$\cap\mu$supp(P).*

Proof :We will only sketch the steps here; see e.g. [1] for a complete treatment. If $(x_0, \xi_0) \notin \mu\text{supp}(P)$, then there is a $Q \in L^m_{1,0}$ such that Q is elliptic at (x_0, ξ_0) and the microsupports of Q and P are disjoint. Then $QP \in L^{-\infty}_{\eta,\delta}$ and $(x_0, \xi_0) \notin \text{WF}(Pu)$. Now assume that $(x_0, \xi_0) \notin \text{WF}(u)$, which implies there is a $Q \in L^m_{1,0}$ such that $Qu \in C^\infty$ and Q is elliptic on a cone around ξ_0. The theorem will follow once we prove the following claim: there are $R, S \in L^m_{1,0}$ such that $(x_0, \xi_0) \notin \text{Char}(R)$ and $RP - SQ \in L^{-\infty}_{\eta,\delta}$. To see that it is sufficient to prove this claim, note that since Q, SQ, and $RP - SQ$ are smoothing, this implies RP is smoothing. It then follows that $(x_0, \xi_0) \notin \text{WF}(Pu)$.□

We can extend the result on elliptic operators to wave front sets by simply replacing the singular support with the wave front set. We then have the following theorem:

Theorem 4.6 *If $PL^m_{\eta,\delta}(\Omega)$ is elliptic, then WF(Pu)=WF(u) $\forall u \in D'(\Omega)$.*

Proof : If Q is a parametrix for P, then by definition $QPu - u \in C^\infty$. Since WF$(QPu - u) = \emptyset$, it is clear that WF(QPu)=WF(u). But by theorem 3.4, we know that WF(Pu) $\subset$WF(u) and WF(u)=WF(QPu) $\subset$WF(Pu). Hence, equality must hold. □

In order to impose a suitable topology on the set of distributions whose wave front set is bounded above, it is necessary to introduce a new collection of seminorms. If $\Gamma \subset T^*X$ is a closed cone, then denote the space of all distributions whose wave front set lies in Γ by $D'_\Gamma(X)$. We say that a sequence of distributions $\{u_j\} \in D'_\Gamma(X)$ converges to $u \in D'_\Gamma(X)$ if $u_j \to u$ weakly *and* $P_{N,\varphi,U}(u_j - u) \to 0$, where P is a seminorm defined by

$$P(u) = \sup_{\xi \in U} |\widehat{\varphi u}|(1 + |\xi|)^N \tag{14}$$

for some $\varphi \in C_c^\infty(X)$ and closed cone $U \subset \mathbf{R}^n$.

The final property of wave front sets that we will give is perhaps the most useful one: defining the product of distributions. The tensor product of two distributions u_1 and u_2 is defined whenever they depend on different sets of variables, but the pointwise product is often ill-defined. We will see that whenever the two wave front sets have a particular form, the product is unambiguously defined. Let $f : X \to Y$ be a smooth map, where X and Y are open subsets of $\mathbf{R}^n$. The pull-back of f, denoted f^*, is defined in the usual way by $f^*(\phi)(x) = \phi(f(x))$.

Definition 4.7 *If $u, v \in D'(\Omega)$, then the product uv is defined to be the pull-back of the tensor product by the diagonal map $\delta : \Omega \to \Omega \times \Omega$ that sends x to (x, x).*

We can now formulate a precise condition for when the product of two distributions is well-defined:

Theorem 4.8 *Let $u_1, u_2 \in D'(X)$, with respective wave front sets WF(u_1), WF(u_2). Then u_1u_2 is defined whenever the composite wave front set WF(u_1)$\oplus$ WF(u_2) $= \{(x, \xi_1 + \xi_2)\,|(x, \xi_1) \in$ WF(u_1); $(x, \xi_2) \in$ WF$(u_2)\}$ does not contain an element of the form $(x, 0)$.*

We conclude this section with examples of this theorem([5]):

i) If $f : U \to V$ is C^∞, then WF$(f) = \emptyset$, and so, as we expect, the product of smooth functions is well-defined.

ii) Let δ_x be the usual delta distribution, i.e. $\delta_x f = f(x)$ for $f \in D(X)$. It is easy to see that WF(δ)=$\{(0,\lambda)\,|\lambda \neq 0\}$. Now we want to consider two delta distributions $\delta(x_1)$ and $\delta(x_2)$. We know that

$$\mathrm{WF}(\delta(x_1)) = \{(0,x_2;\lambda,0)\,|x_2 \in \mathbf{R}, \lambda \neq 0\}$$

$$\mathrm{WF}(\delta(x_2)) = \{(x_1,0;0,\lambda)\,|x_1 \in \mathbf{R}, \lambda \neq 0\}.$$

Since we know that $\lambda \neq 0$, it is clear that $\delta(x_1)\delta(x_2)$ exists and is given by $\delta(x_1,x_2)$ that acts on $f(x_1,x_2)$ by

$$\int f(x_1,x_2)\delta(x_1,x_2)dx_1dx_2 = f(0,0). \tag{15}$$

iii) Let $P(\frac{1}{x})$ be the Cauchy principle part integral given by

$$P(1/x) : f \to \lim_{\epsilon\downarrow 0}\int_{|x|\geq\epsilon}\frac{f(x)}{x}dx. \tag{16}$$

Then for the operator $P(\frac{1}{x}) - i\pi\delta(x)$, we can rewrite it as

$$P(1/x) - i\pi\delta(x) = \lim_{\epsilon\downarrow 0}\frac{1}{x+i\epsilon}. \tag{17}$$

Its wave front set is given by

$$\{(0,\lambda)\,|\lambda > 0\}.$$

By the above theorem, the product of $P(\frac{1}{x})$ with itself clearly exists.

5 Propagation of Singularities

Let (M,ω) be a symplectic manifold, and $H : M \to \mathbf{R}$ a C^r map. We define the Hamiltonian vector field on M generated by H to be the vector field X_H determined by

$$\omega(X_H,\cdot) = dH. \tag{18}$$

Note that the nondegeneracy of ω guarantees the existence of X_H. Due to a theorem of Darboux ([2] p.98), we can always choose local canonical coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ so that

$$\omega = \sum_{i=1}^{n} dq_i \wedge dp_i. \tag{19}$$

It turns out that the principal symbol of a pseudodifferential operator A can also define Hamiltonian vector field in a similar way (A must actually be a special type of operator; see p. 35 of [GS] for details). In the above canonical coordinates, the Hamiltonian field generated by the principal symbol P is

$$X_P = \sum_{i}^{n} \frac{\partial P}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial P}{\partial q_i}\frac{\partial}{\partial p_i}. \tag{20}$$

An integral curve[3] $\gamma : [a, b] \to M$ is called a *bicharacteristic strip* of X_H if H vanishes along γ, i.e. if γ is an integral curve of X_H through $H^{-1}(0)$. We now have assembled the necessary geometrical tools to state the propagation of singularities theorem. The motivation for this result is that since the wave front set is the set of points where $P \in L^m_{1,0}$ is not elliptic, we have that $\mathrm{WF}(u) \subset P^{-1}(0) = \{(x, \xi) \in T^*X \,|\, P(x, \xi) = 0\}$. Our goal is to formulate conditions on certain subsets of $P^{-1}(0)$ so that they are of the form $\mathrm{WF}(u)$ for some u. This idea may be stated precisely as follows:

Theorem 5.1 *Let $A \in L^m_{\eta,\delta}$ and let P_m be its associated real, positive principal symbol of degree m. Furthermore, let $\gamma : [a, b] \to T^*X$ be a bicharacteristic strip of X_{P_m}. If $u \in D'(X)$ satisfies $\gamma([a, b]) \cap WF(Au) = \emptyset$, then either $\gamma([a, b]) \subset WF(u)$ or $\gamma([a, b]) \cap WF(u) = \emptyset$.*

The proof of this theorem is rather involved, and the reader is referred to §8 of [2] for details. The importance of this theorem is that if $A(x, D)u = f$ and (x_0, ξ_0) is a point in $p_A^{-1}(0)$, then the entire bicharacteristic strip beginning at (x_0, ξ_0) is contained in $\mathrm{WF}(u)$. So the singularities of u are literally "flowed" along the vector field generated by p_A.

[3]Recall that an integral curve of a vector field X is a C^k curve γ such that $\gamma'(t) = X(\gamma(t))$ for all t in the domain of γ.

References

[1] G. Folland, *Introduction to Partial Differential Equations*, 2nd edition (Princeton University Press, Princeton, 1995).

[2] A. Grigis and J. Sjostrand, *Microlocal Analysis for Differential Operators* (Cambridge University Press, Cambridge, 1994).

[3] L. Hormander, *The Analysis of Linear Partial Differential Operators I* (Springer-Verlag, Berlin, 1983).

[4] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, Volume I: Functional Analysis* (Academic Press, San Diego, 1980).

[5] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, Volume II: Fourier Analysis and Self-Adjointness*(Academic Press, San Diego, 1974).

[6] M. Taylor, *Pseudodifferential Operators* (Princeton University Press, Princeton, 1981).