MASCHKE'S THEOREM

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1. Introduction

Heinrich Maschke was born on October 24, 1853 in Breslau, Germany. He started his studies at the University of Heidelberg in 1872, studying there under Konigsberger. After a year of military service he continued his studies at the University of Berlin, where he was under the instruction of such outstanding mathematicians as Weierstrass, Kummer and Kronecker. He eventually received his doctorate from Gottingen in 1880. As it was very difficult to get a position as a professor in Germany, Maschke immigrated to the United States. In 1892 at the inception of the University of Chicago department of Mathematics, Eliakim Hastings Moore appointed Maschke as well as Oskar Bolzacore to professorships in the department. The three became the core of the department from 1892 to the time of Maschke's death in 1908.

During his time at Gottingen Maschke became very interested in Klein's ideas on using group theory to solve algebraic equations, and started to work on finite groups of linear transformations. He published Maschke's Theorem in 1899 which states that linear representations of a finite group over fields of characteristic 0, such as the complex, real, and rational numbers, break up into irreducible pieces. A consequence of this theorem is that every FG-module is a direct sum of irreducible FG-submodules, where F is $\mathbb{R}$ or $\mathbb{C}$. This is important because it essentially reduces representation theory to the study of irreducible FG-modules.

In this paper we will state and prove Maschke's Theorem, investigate some of its direct consequences, and explore examples. We will assume that the reader is familiar with basic algebra, but we will define all of the representation theory concepts that are necessary to understand the proof. No new ideas will be presented.

2. Basic Definitions

The basic concept in representation theory is that of a group representation. Let G be a group and let $\mathbf{F}$ be $\mathbb{R}$ or $\mathbb{C}$. Recall that GL$(n, \mathbf{F})$ denotes the group of invertible $n \times n$ matrices with entries in $\mathbf{F}$.

Definition 2.1. A **representation** of G over F is a homomorphism ρ from G to GL$(n, \mathbf{F})$ for some n. The **degree** of ρ is the integer n.

Remark 2.2. We will use the notation of applying homomorphisms on the right, so the image of g under a homomorphism τ is written as $g\tau$. Furthermore by the expression $\tau : g \rightarrow h$ where $g \in$G and $h \in$H, we mean that $h = g\tau$.

Example 2.3. Let G be the dihedral group $\mathrm{D}_8 = \langle$ a, b : $\mathrm{a}^4 = \mathrm{b}^2 = 1$, $\mathrm{b}^{-1}\mathrm{ab} = \mathrm{a}^{-1} \rangle$. Define the matrices A and B by

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and check that

$$A^4 = B^2 = I, \quad B^{-1}AB = A^{-1}$$

It follows that the homomorphism

$$\rho : a^i b^j \longrightarrow A^i B^j \quad (0 \leq i \leq 3, 0 \leq j \leq 1)$$

is a representation of D_8 over $\mathbf{F}$. The degree of ρ is 2.

Next we will define the concept of an FG-module and show the close connection between FG-modules and representations of G over $\mathbf{F}$. This will be useful because we will state and prove Maschke's Theorem in terms of FG-modules.

Definition 2.4. Let V be a vector space over $\mathbf{F}$ and let G be a group. Then V is an **FG-module** if we can define a multiplication vg where v is an element of V, and g is an element of G, satisfying the following conditions for all u,v in V, λ in $\mathbf{F}$ and g,h in G:

(1) $vg \in v$
(2) $v(gh) = (vg)h$
(3) v $1 = v$
(4) $(\lambda v)g = \lambda(vg)$
(5) $(u + v)g = ug + vg$

By conditions (1) (4) and (5) we know that for all $g \in$ G, the function

$$v \longrightarrow vg \quad (v \in V)$$

is an endomorphism of V.

Definition 2.5. Let V be an FG-module. A subset W of V is said to be an **FG-submodule** of V if W is a subspace and for all $w \in$W and all $g \in$ G$wg \in$ W, i.e. an FG- submodule of V is a subspace which is also an FG-module.

Clearly for every FG-module V, the zero subspace, denoted 0 and V itself are FG-submodules of V.

Example 2.6. Let G=$C_2 = \langle a : a^2 = 1\rangle$ and let V = $\mathbf{F}^2$ where $\mathbf{F}$ is either $\mathbb{R}$ or $\mathbb{C}$. For $(\alpha, \beta) \in V$ define $(\alpha, \beta)1$=(α, β) and $(\alpha, \beta)\alpha$=(β, α). It is clear that V is an FG-module because (α, β) and (β, α) are both linear combinations of (α, β) and $(\alpha, \beta) \in V$. Does V have any FG-submodules? Assume yes. Let U be a non-zero FG-submodule of V, and let $(\alpha, \beta) \in U$ with $(\alpha, \beta) \neq (0, 0)$. Then $(\alpha, \beta) + (\alpha, \beta)\alpha = (\alpha + \beta, \alpha + \beta) \in U$, and $(\alpha, \beta) - (\alpha, \beta)\alpha = (\alpha - \beta, \beta - \alpha) \in U$. Now at least one of $\alpha + \beta$ and $\alpha - \beta$ is nonzero. Thus (1,1) or (1, -1) belongs to U, so the FG-submodules of V are 0, sp((1,1)), sp((1, -1)) and V. This example comes from the exercises in [2].

Definition 2.7. Let V and W be FG-modules. A function τ:V $\rightarrow$ W is said to be an **FG-homomorphism** if τ is a linear transformation and for all $v\in$ V, $g\in$G$(vg)\tau =(v\tau)g$. That is to say, if τ sends v to w then it sends vg to wg.

FG-homomorphisms are the structure-preserving functions for FG-modules in the same way that group homomorphisms are to groups and linear transformations are to vector spaces. Now we shall see that FG-homomorphisms give rise to FG-submodules in a natural way.

Lemma 2.8. *Let V and W be FG-modules and let τ:V $\rightarrow$ W be an FG-homomorphism. Then Ker(τ) is an FG-submodule of V and Im(τ) is an FG-submodule of W.*

Proof. Clearly Ker (τ) is a subspace of V and Im τ is a subspace of W, since τ is a linear transformation. Let $v \in$ Ker(τ) and $g \in$ G. Then

$$\begin{aligned} (vg)\tau &= (v\tau)g \\ &= 0g \\ &= 0 \end{aligned}$$

so $vg \in$Ker(τ). Thus Ker(τ) is an FG-submodule of V. Put $w \in$ Im (τ) to that w=$v\tau$ for some $v\in$ V. Then for all $g\in$G

$$wg \;=\; (v\tau)g \;=\; (vg)\tau \in Im(\tau)$$

so Im(τ) is an FG-submodule of W. □

We need two more definitions that will be useful later.

Definition 2.9. Let G be a subgroup of S_n. The FG-module V with basis $v_1, ..., v_n$ such that v_i g=v_{ig} for all i, and all $g \in$ G is called the **permutation module** for G over F. We call $v_1, ..., v_n$ the **natural basis** of V.

Definition 2.10. Let V be an FG-module, and let B be a basis of V. For each $g \in$ G let $[g]_B$ denote the matrix of the endomorphism $v \rightarrow vg$ of V relative to the basis B.

3. Maschke's Theorem

Theorem 3.1. *Let G be a finite group, let* F *be* $\mathbb{R}$ *or* $\mathbb{C}$*, and let V be an FG-module. If U is an FG-submodule of V, then there is an FG-submodule W of V such that V=$U \oplus W$.*

Proof. First choose any subspace W_0 of V such that V=U$\oplus W_0$. We can find such a W_0 by taking a basis $v_1, ..., v_m$ of U, and extending it to a basis $v_1, ..., v_n$ of V. Then $W_0 =$ sp$(v_{m+1}, ..., v_n)$.

Now for all $v \in$ V there exists unique vectors $u \in$ U and $w \in W_0$ such that $v = u + w$. We define $\phi : V \to V$ by setting $v\phi$=u. Recall from algebra that if V=U $\oplus$ W, and if we define π: V $\to$ V by

$$(u + w)\pi = u \quad \forall u \in U, w \in W$$

then π is an endomorphism of V. Moreover, Im(π) =U, Ker(π) = W and π^2= π. With this in mind we see that ϕ is a projection of V with kernel W_0 and image U. Our aim is to modify the projection ϕ to create an FG-homomorphism from V $\to$ V with image U. Define τ: V $\to$ V by

$$v\tau = \frac{1}{|G|} \sum_{g \in G} vg\phi g^{-1} \quad v \in G \tag{1}$$

Then τ is an endomorphism of V and Im$(\tau) \subseteq$ U.

Now we will show that τ is an FG-homomorphism. For $v \in$ V and $x \in$ G we have

$$vx\tau = \frac{1}{|G|} \sum_{g \in G} (vx)g\phi g^{-1}$$

As g runs through the elements of G, so does $h = xg$. Thus we have

$$\begin{aligned} (vx)\tau &= \frac{1}{|G|} \sum_{h \in G} vh\phi h^{-1}x \\ &= \left(\frac{1}{|G|} \sum_{h \in G} vh\phi h^{-1} \right) x \\ &= (v\tau)x \end{aligned}$$

Thus τ is an FG-homomorphism.

It remains to show that τ is a projection with image U. To show that τ is a projection it suffices to demonstrate that $\tau^2 = \tau$. Note that given $u\in$ U and $g\in$ G, we have $ug\in$ U, so $(ug)\phi{=}ug$. Using this we see that:

$$\begin{aligned} u\tau &= \frac{1}{|G|}\sum_{g\in G} ug\phi g^{-1} \\ &= \frac{1}{|G|}\sum_{g\in G}(ug)g^{-1} \\ &= \frac{1}{|G|}\sum_{g\in G} u \\ &= u \end{aligned}$$

Now let $v\in$ V. Then $v\tau\in$U, so we have $(v\tau)\tau = v\tau$. We have shown that $\tau^2 = \tau$.

Summary . We have established that τ: V $\to$ V is a projection and an FG-homomorphism. Furthermore Im(τ) =U. If we let W= Ker(τ) then W is an FG-submodule of V and V=U$\oplus$W. So we are done. The ideas in this proof are taken from [2].

□

Remark 3.2. The converse of Maschke's Theorem is also true. That is, if the characteristic of F does not divide $|G|$, then G possesses finitely generated FG-modules which are not completely reducible. Specifically, the FG-module itself is not completely reducible.

We can also state the theorem in more generality. Let G be a finite group and let F be a field whose characteristic does not divide $|G|$. If V is any FG-module and U is any submodule of V, then V has a submodule W such that V=U$\oplus$W. Observe that the hypothesis of Maschke's Theorem applies to any finite group when F has characteristic 0.

A full proof of the more general statement appears in [1] but we will outline it. We aim to produce an FG-module homomorphism $\pi : V \to U$ which is a projection onto U, such that it satisfies the following two properties:

$$\text{i)} \qquad (u)\pi = u \text{ for all } u \in U$$

$$\text{ii)} \qquad ((v)\pi)\pi = (v)\pi \text{ for all } v \in V$$

Note that the second property is implied by the first and the fact that (V)π $\subseteq$U. We assume that we can produce such an FG-module homomorphism. Then the kernel of that homomorphism (say W=Ker(π)) is the direct sum complement to U. Thus U$\cap$W=0, and V=U$\oplus$W. Then we just have to find such an FG-module projection π.

Since U is a subspace it has a vector space direct sum compliment W_0 in V, which we can find as we did above, by taking a basis for U and building up to a basis for V. Thus V=U$\oplus W_0$ as vector spaces, but W_0 need not be G-stable. We let π_0:V$\rightarrow$U be the vector space projection of V onto U associated with this direct sum decomposition. Thus

$$(u+w)\pi_0 = u \quad \text{for all } u \in U \text{ and } w \in W$$

The key is to 'average' π_0 over G to form an FG-module projection π. We do this similarly to the procedure above.

Let n=$|G|$ and view n as an element of F, such that n=(1+1+1..., n times). By hypothesis n is not zero in F so it has an inverse in F. We define

$$\pi = \frac{1}{n}\sum_{g \in G} g\pi_0 g^{-1}$$

Then

(1) $\pi : V \rightarrow U$ is a linear transformation

(2) $(u)\pi \;=\; u$ for all $u \in$ U

(3) $(v)\pi^2 = (v)\pi$

It remains to show that π is an FG-module homomorphism. It suffices to prove that for all $h \in$ G, $(hv)\pi$=$(h\pi)v$ for $v \in$ V, which we do in the same way as above.

Example 3.3. This example comes from [2], and illustrates the matrix version of Maschke's Theorem. Namely, suppose that ρ is a reducible representation of a finite group G over F of degree n. Then we know that ρ is equivalent to a representation where for all $g \in G$, $g \rightarrow$

$$\left|\begin{array}{c|c} X_g & 0 \\ \hline Y_g & Z_g \end{array}\right|$$

for some matrices X_g, Y_g, Z_g where X_g is $k \times k$ with $0 < k < n$. Using Maschke's Theorem we can go a step further to say that ρ is equivalent to a representation where $g \rightarrow$

$$\left|\begin{array}{c|c} A_g & 0 \\ \hline 0 & B_g \end{array}\right|$$

where A_g is also a $k \times k$ matrix.

Let G=S_3 and let V=sp(v_1, v_2, v_3) be the permutation module for G over F. Say $u = v_1+v_2+v_3$ and U=sp(u). Then U is the FG-submodule of V since ug=u for all $g \in$ G. We use the proof of Maschke's Theorem to find an FG-submodule W of V such that V=U$\oplus$W. To begin we let W_0=sp(v_1, v_2). Then V=U$\oplus W_0$, but W_0 is not an FG-submodule. The projection ϕ onto U is given by

$$\phi : v_1 \rightarrow 0, \quad v_2 \rightarrow 0, \quad v_3 \rightarrow v_1 + v_2 + v_3$$

Now we can check that that FG-homomorphism τ given by equation 1 is

$$\tau : v_1 \rightarrow \frac{1}{3}(v_1 + v_2 + v_3) \ \ (i = 1, 2, 3)$$

The required FG-submodule W is then Ker(τ), so

$$W = sp(v_1 - v_2, v_2 - v_3)$$

Observe that if B is the basis $v_1 + v_2 + v_3, v_1, v_2$ of V, then for all $g \in$ G the matrix $[g]_B$ has the form

$$[g]_B = \begin{pmatrix} * & 0 & 0 \\ * & * & * \\ * & * & * \end{pmatrix}$$

where the zeros reflect the fact that U is an FG-submodule of V. If we use $v_1+v_2+v_3, v_1-v_2, v_2-v_3$ instead as a basis B' the we get a matrix of the form

$$[g]_{B'} = \begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$$

because $\mathrm{sp}(v_1 - v_2, v_2 - v_3)$ is also an FG-submodule of V.

4. Conclusion

Maschke's Theorem is a fundamental result in representation theory. It is because of Maschke's Theorem that we can conclude that every non-zero FG-module is a direct sum of irreducible FG-submodules. Furthermore, we can use Maschke's Theorem to prove that if V is an FG-module where F is $\mathbb{R}$ or $\mathbb{C}$, G is a finite group, and U is an FG-submodule of V, then there exists a surjective FG-homomorphism from V onto U. The consequences of Maschke's Theorem are far reaching, and nontrivial.

References

[1] David S Dummit and Richard M Foote. *Abstract Algebra*. John Wiley & Sons, Inc, New York, 2004.

[2] Gordon James and Martin Liebeck. *Representations and Characters of Groups*. Cambridge University Press, New York, 2001.