Problems in Ordinary Differential Equations

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Chapter 1

Peano Uniqueness Theorem

Exercise *(Peano Uniqueness Theorem)* For each fixed x, let $F(x,y)$ be a non-increasing function of y. Show that, if $f(x)$ and $g(x)$ are two solutions to $y' = F(x,y)$, and $b > a$, then $|\,f(b) - g(b)\,| \leq |\,f(a) - g(a)\,|$. Use this fact to infer a uniqueness theorem.

1.1 Preliminaries

Definition 1 A solution, f, to a differential equation is called *unique* if it is the only solution to a differential equation, up to constants appearing in the solution f. The idea here is that the solution is unique given some initial condition, $y_0 = f(x_0)$; this initial condition allows us to determine the constants that may appear in f.

Example 1 $y = C \cdot e^{k \cdot x}$ is the unique solution to the differential equation $y' = k \cdot y$, even though the constant C allows for more than one solution. However, given some initial condition, the solution is unique. The derivation is commonly known, so I will omit it here.

Definition 2 [1] A function, $F(x,y)$, is said to satisfy a *one-sided Lipschitz condition* in a domain D if, for some finite constant, L,

$$y_i > y_j \Rightarrow F(x, y_i) - F(x, y_j) \leq L \cdot (y_i - y_j) \tag{1.1}$$

[1] Birkhoff, Garrett and Rota, Gian-Carlo. Ordinary Differential Equations. Fourth Edition. 1989. Hoboken, New Jersey.

I also find it useful to define a *Lipschitz condition* in the same domain, D, as when the same function, $F(x, y)$, satisfies the condition that $\exists L \geq 0$ such that

$$|\,F(x,y) - F(x,z)\,| \leq \; L\cdot|\,y - z\,| \tag{1.2}$$

Lemma 1 [2] Let $\sigma(x)$ be a differentiable function satisfying the differential inequality

$$\sigma'(x) \;\leq\; K \cdot \sigma(x) \quad for \quad a \;\leq\; x \;\leq\; b \tag{1.3}$$

where K is a constant. Then

$$\sigma(x) \;\leq\; \sigma(a) \cdot e^{K\cdot(x-a)} \quad for \quad a \;\leq\; x \;\leq\; b \tag{1.4}$$

Proof. Multiply both sides of (1.3) by $e^{-K\cdot x}$ and transpose to obtain

$$e^{-K\cdot x} \cdot [\sigma'(x) - K \cdot \sigma(x)] \;\leq\; 0$$

But, I note that the left side of this equation is equal to $\frac{d}{dx}\left[\sigma(x) \cdot e^{-K\cdot x}\right]$; differentiation (with the product rule) will easily confirm this. Thus, the function $\sigma(x) \cdot e^{-K\cdot x}$ has a negative or zero derivative and is thus non-increasing for $a \;\leq\; x \;\leq\; b$. Thus, I find that $\sigma(x) \cdot e^{-K\cdot x} \;\leq\; \sigma(a) \cdot e^{-K\cdot a}$. [3] $\diamond$

Lemma 2 [4] The one-sided Lipschitz condition, (1.1) implies that

$$[g(x) - f(x)] \cdot [g'(x) - f'(x)] \;\leq\; L \cdot [g(x) - f(x)]^2 \tag{1.5}$$

for any two solutions $f(x)$ and $g(x)$ of $y' = F(x, y)$.

Proof. Setting $f(x) = y_1$, $g(x) = y_2$, I have

$$[g(x) - f(z)] \cdot [g'(x) - f'(x)] = (y_2 - y_1) \cdot [F(x, y_2) - F(x, y_1)]$$

from the differential equation given. If $y_2 > y_1$, then, by (1.2), the right side of this equation has the upper bound $L \cdot (y_2 - y_1)^2$. Noting that y_1 and y_2 can be interchanged without any effect on the above expressions, I have proven Lemma 2. [5] $\diamond$

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[2] Ibid.
[3] Ibid.
[4] Ibid.
[5] Ibid.

Definition 3 [6] A differential equation is said to be *normal* if it is in the form

$$y = f(x, y) \tag{1.6}$$

Theorem 1 [7] Let $f(x)$ and $g(x)$ be any two solutions of the first-order normal differential equation $y' = F(x, y)$ in a domain, D, where F satisfies the one-sided Lipschitz condition (1.1). Then,

$$|\,f(x) - g(x)\,| \leq \; e^{L\cdot(x-a)} \cdot |\,f(a) - g(a)\,| \quad if \quad x > a \tag{1.7}$$

Proof. Consider the function

$$\vartheta(x) = [g(x) - f(x)]^2.$$

Computing its derivative with elementary calculus, I find that

$$\vartheta'(x) = 2 \cdot [g(x) - f(x)] \cdot [g'(x) - f'(x)].$$

By Lemma 2, this implies that $\vartheta'(x) \leq 2 \cdot L \cdot \vartheta(x)$; by Lemma 1, this implies $\vartheta(x) \leq e^{2\cdot L\cdot(x-a)} \cdot \vartheta(a)$. Taking the square root of both sides of this inequality (both of which I know to be non-negative), I get (1.7), thus completing the proof. [8] $\diamond$

1.2 Solution to Exercise

For those of us with very short memories, I'll restate the exercise:

Exercise *(Peano Uniqueness Theorem)* For each fixed x, let $F(x, y)$ be a non-increasing function of y. Show that, if $f(x)$ and $g(x)$ are two solutions to $y' = F(x, y)$, and $b > a$, then $|\,f(b) - g(b)\,| \leq |\,f(a) - g(a)\,|$. Use this fact to infer a uniqueness theorem.

[6] Ibid.
[7] Ibid.
[8] Ibid.

Solution For fixed x, I know that $F(x, y)$ is non-increasing. Thus, taking $y_2 > y_1$, I immediately find that $F(x, y_2) \le F(x, y_1)$. Thus, I can infer

$$F(x, y_2) - F(x, y_1) \le 0 = 0 \cdot (y_2 - y_1)$$

which tells us that for fixed x, $F(x, y)$ satisfies the one-sided Lipschitz inequality, (1.1). I can thus apply Theorem 1, letting $x = b$,

$$| f(b) - g(b) | \le e^{0 \cdot (y_2 - y_1)} \cdot | f(a) - g(a) | = | f(a) - g(a) |$$

Since this immediately places an upper bound on the amount that any two solutions to the differential equation $y' = F(x, y)$ can differ from one another, and, if I define another function $h(x) =| f(x) - g(x) |$ for $x > a$, then $h(a) = L$ and $h(x)$ is a non-increasing function, so, $h(x) \in [0, L]$ for some $L \ge 0$. And, since uniqueness is only up to the initial given condition, if I assume that $g(a) = f(a)$, then immediately $L = 0$, and hence $h(x) = 0 \, \forall x > a$. Thus, I can conclude that if any two solutions have the same initial condition, then they are precisely the same solution. $\diamond$

Chapter 2

Peano Existence Theorem

Exercise *(Peano Existence Theorem)* If the function $\mathbf{X}(\mathbf{x}, \mathbf{t})$ is continuous for $|\,\mathbf{x} - \mathbf{c}\,| \leq \delta$, and $|\,t - a\,| \leq \gamma$, and if $|\,\mathbf{X}(\mathbf{x}, \mathbf{t})\,| \leq \theta$ there, then the vector differential equation,

$$\frac{dx}{dt} = \mathbf{X}(\mathbf{x}, \mathbf{t}) \quad \textbf{or} \quad \mathbf{x}'(\mathbf{t}) = \mathbf{X}(\mathbf{x}, \mathbf{t}) \tag{2.1}$$

has at least one solution, $\mathbf{x}(\mathbf{t})$ defined for

$$|\,t - a\,| \leq min(\gamma, \frac{\delta}{\theta})$$

and satisfying the initial condition $\mathbf{x}(\mathbf{0}) = \mathbf{c}$.

2.1 Preliminaries

I would like to discuss this problem in the context of Banach Spaces, since this will clairify the nature of our main theorem in this chapter. I introduce these main ideas below:

Definition 1 [1] A complex vector space, X is said to be a *normed linear space* if to each $x \in X$ there is associated a non-negative real number, $\|x\|$, called the *norm* of x, such that

(a) $\|x + y\| \leq \|x\| + \|y\|$,

(b) $\|\alpha x\| = |\alpha| \cdot \|x\|$,

[1] Rudin, Walter. Real and Complex Analysis. Third Edition. 1987. McGraw-Hill. Boston.

(c) $\|x\| = 0 \Leftrightarrow x = 0$.

by (a), I can deduce (trivially) the *Triangle Inequality*

$$\|x - y\| \leq \|x - z\| + \|z - y\| \quad x, y, z \in X. \tag{2.2}$$

Definition 2 A *Metric Space* is a normed linear space with a function, $\xi : X \times X \to \mathbf{R}_{\geq 0}$. The function ξ satisfies the following properties:

(a) $\xi(x, y) = 0 \iff x = y$

(b) ξ satisfies the triangle inequality (note that this is the same as (2.2) in the case of a normed linear space),

$$\xi(x, y) \leq \xi(x, z) + \xi(z, y)$$

(c) $\xi(x, y) = \xi(y, x)$

Definition 3 A sequence of terms in a metric space, (x_n) is said to be *Cauchy* if $\forall \epsilon > 0, \exists n \in \mathbf{N}$ such that $\forall m, s \geq n \Rightarrow \xi(x_n, x_m) < \epsilon$.

Definition 4 A Metric Space, X, is said to be *Complete* if every Cauchy sequence in X converges in X. e.g. for all such (x_n) as above $\exists \varsigma \in X$ such that $\forall \epsilon > 0, \exists n \in \mathbf{N}$ such that $m > n \Rightarrow \xi(x_m, \varsigma) < \epsilon$.

Definition 5 [2] A *Banach Space* is a normed linear space which is complete with respect to the induced metric (as the normed linear space induced a metric with the norm, in (2.2) as compared to the definition of the metric in the metric space).

Definition 6 A *measure* is a function $\mu : \mathcal{P}(X) \to \mathbf{R}_{\geq 0}$ such that

1 $\mu(\emptyset) = 0$

2 If $E_1, E_2, E_3, \ldots$ form a countably additive set of subsets of X such that $i \neq j \Rightarrow E_i \cap E_j = \emptyset$ then $\mu(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i)$.

Definition 7 We now define the $L^p(\mu)$-spaces (or just L^p-spaces for short).

[2] Ibid.

This is the class of Lebesgue-integerable functions (with respect to the given measure, μ). We formally define

$$L^p(\mu) = \Big\{ f : X \to S \mid \big(\int_X |f|^p \cdot d\mu \big)^{1/p} < \infty \Big\}.$$

It may be easier to consider $L^1(\mu)$, thinking of μ as the standard Euclidian measure on $\mathbf{R}^n$ (length). I do note, though, that functions in $L^p(\mu)$ are only defined up to their equivalence class. Two functions are in the same equivalence class if they differ on a set of measure zero (interestingly, $\mathbf{Q}$ has measure zero in $\mathbf{R}$ with respect to the standard measure). As it is defined below, this would be equivalent to saying that $\xi(f, g) = 0$. This does not, however, imply that $f \equiv g$. It is easy to show that this equivalence relation is well defined, so I will not do it here (for a more in-depth discussion of measure and Lebesgue spaces, consult Rudin's Principles of Mathematical Analysis 1964).

Since I am concerned with a continuous function on a compact set, it is clear that if I am considering the function on X, any compact set, then the function is in $L^1(X)$. Furthermore, if Γ is the class of all continuous functions on X, then $\Gamma \subseteq L^1(X)$, which I know to be a Banach Space, since $L^p(\mu)$ is a Banach Space for all p (for a greater consultation of this fact, consider Rudin's Real and Complex Analysis, 1987).

Conveniently, since I desire to use the integral norm, $\|\cdot\|_p$ such that

$$\xi_p(f, g) = \|f - g\|_p = \Big(\int_X |f - g| d\mu \Big)^{1/p}$$

As with $L^p(\mu)$, I restrict myself to considering functions, f, for which $\|f\|_p < \infty$. (For our purposes here, $p = 1$.)

Definition 8 [3] A family, Ω, of vector-valued functions $f_n(t)$, defined on an interval, $\mathbf{I}$, is said to be *equicontinuous* when, given $\epsilon > 0$, $\exists \delta > 0$ such that

$$|t - s| < \delta \Rightarrow |\mathbf{x}(t) - \mathbf{x}(s)| < \epsilon \quad \text{for all } \mathbf{x} \in \Omega \text{ provided that } s, t \in \mathbf{I}$$

[3] Birkhoff, Garrett and Rota, Gian-Carlo. Ordinary Differential Equations. Fourth Edition. 1989. Hoboken, New Jersey.

Theorem 1 [4] *(Arzela-Ascoli Theorem)*

1 Let $f_n(t)$, $(n \in \mathbf{N})$, be a bounded equicontinuous sequence of scalar or vector functions, defined for $a \leq t \leq b$. Then there exists a subsequence, f_{n_i} that is uniformly convergent in the interval. Or, equivalently,

2 Suppose that Ω is a pointwise bounded equicontinuous collection of complex functions on a metric space, X, and that X contains a countable dense subset, E.
Every sequence, $\{f_n\}$ in Ω has then a subsequence that converges uniformly on every compact subset of X (This is a somewhat stronger statement; I will prove the stronger statement, and only use the weaker statement).

Proof.

1 Let $x_1, x_2, x_3, x_4, \ldots$ be an enumeration of the points of E. Let S_0 be the set of all positive integers (I will think of S_m as some subset of all the positive integers strictly greater than m). Suppose $k \geq 1$ and an infinite set, $S_{k-1} \subset S_0$ has been chosen. Since $\{f_n(x_k) \mid n \in S_{k-1}\}$ is a bounded sequence of complex numbers, it has a convergent subsequence (since $mathbfC$ is complete). In other words, there is an infinite set, $S_r \subset S_{k-1}$ so that lim $f_n(x_k)$ exists as $n \to \infty$ within S_r.

2 Continuing in this way, I obtain infinite sets $S_0 \supset S_1 \supset S_2 \supset \cdots$ with the property that *lim* $f_n(x_j)$ exists for $1 \leq j \leq k$ if $n \to \infty$ within S_r.

3 Let ω_k be the k^{th} term of S_r (with respect to the natural order of the positive integers) and put

$$S = \{\omega_1, \omega_2, \omega_3, \ldots\}.$$

For each r there are then at most $r-1$ terms of S that are not in S_r.

4 Hence *lim* $f_n(x)$ exists, for every $x \in E$ as $n \to \infty$ within S.

[4] Rudin, Walter. Real and Complex Analysis. Third Edition. 1987. McGraw-Hill. Boston.

5 Now let $K \subset X$ be compact, and pick any $\epsilon > 0$. By equicontinuity, there is a $\delta > 0$ such that $\xi_1(p,q) < \delta \Rightarrow |f_n(p) - f_n(q)| < \epsilon \; \forall n$. Cover K with open balls, $B_1, B_2, B_3, \ldots B_M$ of radius $\frac{\delta}{2}$. Since E is dense in X, there are points $p_i \in \{E \bigcap B_i\}$ for $1 \leq i \leq M$. Since $p_i \in E$, $lim\ f_n(p_i)$ exists as $n \to \infty$ within S. Hence there is an integer N such that

$$|f_m(p_i) - f_n(p_i)| \leq \epsilon$$

for $i = 1, 2, \ldots M$. If $m > N, n > N$, and m and n are in S.

6 To finish, pick $x \in K$. Then $x \in B_i$ for some i, and $\xi(x, p_i) < \delta$. Our choice of δ and N show that

$$|f_m(x) - f_n(x)| \leq |f_m(x) - f_m(p_i)| + |f_m(p_i) - f_n([_i)| + |f_n([_i) - f_n(x)| < \epsilon + \epsilon + \epsilon = 3 \cdot \epsilon$$

assuming that $m > N$, $n > N$, $m \in S$, $n \in S$. [5] $\diamond$

2.2 Proof of Peano Existence Theorem

Recall,
Theorem 2 *(Peano Existence Theorem)* If the function $\mathbf{X}(\mathbf{x}, \mathbf{t})$ is continuous for $|\,\mathbf{x} - \mathbf{c}\,| \leq \delta$, and $|\,t - a\,| \leq \gamma$, and if $|\,\mathbf{X}(\mathbf{x}, \mathbf{t})\,| \leq \theta$ there, then the vector differential equation,

$$\frac{dx}{dt} = \mathbf{X}(\mathbf{x}, \mathbf{t}) \quad \textbf{or} \quad \mathbf{x}'(\mathbf{t}) = \mathbf{X}(\mathbf{x}, \mathbf{t}) \tag{2.3}$$

has at least one solution, $\mathbf{x}(\mathbf{t})$ defined for

$$|\,t - a\,| \leq min(\gamma, \frac{\delta}{\theta})$$

and satisfying the initial condition $\mathbf{x}(\mathbf{0}) = \mathbf{c}$.

Proof. First, I note that I can consider the equivalent expression to equation (2.3),

$$x(t) = c + \int_a^t \mathbf{X}(\mathbf{x}(s), s) \cdot ds. \tag{2.4}$$

[5] Ibid.

Clearly, (2.3) has a solution $\iff$ (2.4) has a solution.
Now, I desire to construct a sequence of functions, (f_n), that will converge to the desired function $x(t)$. However, when I consider the equivalent statement to (2.4) for each element of the sequence of functions, it must be integrable. Without loss of generality, I may assume that $a = 0$, since this will not change anything fundamental, but will only make everything appear neater (so I don't account for phase shifts). Now, I let $\Delta = min(\gamma, \frac{\delta}{\theta})$. I will construct our sequence of functions to be defined on the interval $[0, \Delta]$. For each n, I define f_n by

$$f_n(t) = \begin{pmatrix} c & \text{if } t \in [0, \frac{\Delta}{n}] \\ c + \int_0^{t-\frac{\Delta}{n}} \mathbf{X}(f_n(s), s) \cdot ds & \text{if } t \in (\frac{\Delta}{n}, \Delta] \end{pmatrix}$$

Clearly, f_n is continuous for each n, and as $n \to \infty$, $f_n(t) \to x(t)$
Now, I note that this sequence of functions is uniformly bounded, e.g.,

$$|f_n(t)| \leq |c| + \int_0^{\Delta} \theta \cdot ds \leq |c| + \Delta \cdot \theta. \tag{2.5}$$

I will now make use of the following inequality, which can be easily derived from the triangle inequality (above), but whose derivation I will not show here.

$$| \int_a^b f(t)dt \mid \leq \int_a^b |f(t)| \cdot dt \tag{2.6}$$

Now, combining (2.5) and (2.6), I can form the resulting equation,

$$|f_n(t_2) - f_n(t_1)| \leq \int_{t_1-\frac{\Delta}{n}}^{t_2-\frac{\Delta}{n}} |\mathbf{X}(f_n(s), s)| \cdot ds \leq \theta \cdot |t_2 - t_1| \tag{2.7}$$

From (2.7), it is clear that the $f_n(t)$ are equicontinuous.
Now, I apply Theorem 1 (Arzela-Ascoli), since all of the conditions of the theorem are satisfied. Thus, I find that there is at least one solution to equation (2.4), which completes the proof. $\diamond$

Chapter 3

Poincaré-Bendixson

3.1 Preliminaries

Definition 1 [1] A point y_α is called a *stationary* or *singular* point of

$$y' = f(y) \tag{3.1}$$

if $f(y_\alpha) = 0$. If $f(y_\alpha) \neq 0$, then the point y_α is called a *regular* point.

Definition 2 [2] If (3.1) has a solution that is defined on the half plane, $t \geq t_0$, then I denote the set of points that are a solution to (3.1) by C^+ indicating $y = y(t),\ t \geq t_0$. Now, I denote by $\Omega(C^+)$ the set of points y_α such that there exists a sequence $t_0 < t_1 < \ldots$ such that $t_n \to \infty$ and $y(t_n) \to y_\alpha$ as $n \to \infty$. The analogue in the other direction (as $t_n \to -\infty$) is the set $A(C^+)$. Similarly, if $y = y(t)$ is defined on $-\infty < t < \infty$, its set of limit points is defined to be $A(C^+) \cup \Omega(C^+)$.

Theorem 1 [3] Assume that $f(y)$ is continuous on an open y-set E and that $C^+ : y = y_\beta(t)$ is a solution of (3.1) for $t \geq t_0$. Then $\Omega(C^+)$ is closed. If C^+ has a compact closure in E, then $\Omega(C^+)$ is connected.

Proof. $\Omega(C^+)$ is closed trivially. The content of the theorem really lies in the second half of the statement. Now, $\Omega(C^+)$ is contained in the closure of the set of points $C^+ : y = y_\beta(t),\ t \geq t_0$. This implies that $\Omega(C^+)$ is

[1]Hartman, Philip. Ordinary Differential Equations. John Wiley and Sons, Inc.. New York. 1964

[2]Ibid.

[3]Ibid.

compact. Now, by way of contradiction, I assume that $\Omega(C^+)$ is not connected, hence it has a decomposition into a disjoint union of two closed sets (and hence compact since they are subsets of $\Omega(C^+)$), C_1 and C_2 such that $dist(C_1, C_2) = \delta > 0$. There exists a sequence, (t_n) with $t_0 < t_1 < \dots$ satisfying $dist(y_\beta(t_{2n+1}), C_1) \to 0$ and $dist(y_\beta(t_{2n}), C_2) \to 0$ as $n \to \infty$. Hence, for large n, there is a point, $t = t_n*$ such that $t_n < t_n* < t_{n+1}$ and $dist(y_\beta(t_n*), C_i) \geq \frac{\delta}{4}$ for $i = 1, 2$. The sequence $y_\beta(t_1*), y_\beta(t_2*), y_\beta(t_3*), \dots$ has a cluster point y_γ, since C^+ has compact closure. Clearly, $y_\beta \subset \Omega(C^+)$ and $dist(y_\beta, C_i) \geq \frac{\delta}{4}$ for $i = 1, 2$. This contradiction proves the theorem. [4] $\diamond$

Now I will present two results which I will not prove. However, for a quick, painless proof of both of these, see Hartman, Philip. Ordinary Differential Equations [1964], pages 12-15.

Theorem 2 Let $f(t, y)$ be continuous on an open (t, y)-set, E, and let $y(t)$ be a solution of $y' = f(t, y)$ on some interval. Then $y(t)$ can be extended (as a solution) over a maximal interval of existence, (θ_-, θ_+). Also, if (θ_-, θ_+) is a maximal interval of existence, then $y(t)$ tends to the boundary ∂E of E as $t \to \theta_-$ and $t \to \theta_+$.

Theorem 3 Let $f(t, y)$ and $f_1(t, y), f_2(t, y), f_3(t, y), \dots$ be a sequence of continuous functions defined on an open (t, y)-set, E, such that $f_n(t, y) \to f(t, y)$ as $n \to \infty$ holds uniformly on every compact subset of E. Let $y_n(t)$ be a solution of $y' = f_n(t, y)$, $y(t_n) = y_{n0}$ for $(t_n, y_{n0}) \in E$, and let $(\theta_{n-}, \theta_{n+})$ be its maximal interval of existence. Let $(t_n, y_{n0}) \to (t_0, y_0) \in E$ as $n \to \infty$. Then there is a solution, $y(t)$ of $y' = f(t, y)$, $y(t_0) = y_0$, having a maximal interval of existence, (θ_-, θ_+), and a sequence of positive integers, $n_1 < n_2 < \dots$ with the property that if $\theta_- < t_1 < t_2 < \theta_+$, then $\theta_{n-} < t_1 < t_2 < \theta_{n+}$ for $n = n_k$ and k large, and $y_{n_k}(t) \to y(t)$ as $k \to \infty$ uniformly for $t \in [t_1, t_2]$. In particular, $limsup\, \theta_{n-} < \theta_- < \theta_+ < liminf\, \theta_{n+}$ as $n = n_k \to \infty$.

Theorem 4 [5] Assume that $f(y)$ is continuous on an open y-set E and that $C^+ : y = y_\beta(t)$ is a solution of (3.1) for $t \geq t_0$. Assume also that $y_\alpha \in E \cap \Omega(C^+)$. Then

$$y' = f(y), \quad y(0) = y_\alpha \tag{3.2}$$

[4] Ibid.
[5] Ibid.

has at least one solution $y = y_\alpha(t)$ on a maximal interval (θ_-, θ_+) such that $y_\alpha(t) \in \Omega(C^+)$ for $t \in (\theta_-, \theta_+)$. In particular, when C^+ has a compact closure in E, then $C_\alpha : y = y_\alpha(t)$ exists on $(-\infty, \infty)$ and $C_\alpha \cup A(C_\alpha) \cup \Omega(C_\alpha) \subset \Omega(C^+)$.

Proof. Let $t_0 < t_1 < \ldots$ and $t_n \to \infty$, $y_n \to y_\alpha$ as $n \to \infty$, where $y_n = y_\beta(t_n)$. Then $y_n(t) = y_\beta(t + t_n)$ is a solution of

$$y' = f(y) \quad y(0) = y_n. \tag{3.3}$$

Using Theorem 3, above, letting $f_n(t, y) = f(y)$ for $n = 1, 2, 3, \ldots$, such that I can assume the existence of a solution to (3.2), $y_\alpha(t)$ on a maximal interval (θ_-, θ_+). Also, there exists a sequence of positive integers, $n_1 < n_2 < n_3 < \ldots$ such that

$$y_\alpha(t) = \lim_{k\to\infty} y_{n_k}(t) = \lim_{k\to\infty} y_\beta(t + t_{n_k}) \tag{3.4}$$

holds uniformly on compact subintervals of (θ_-, θ_+). Also, it is easily seen that $y_\alpha(t) \in \Omega(C^+)$ for $t \in (\theta_-, \theta_+)$. This proves the first part of the theorem.

The second part concerning existence on $(-\infty, \infty)$ follows at once from Theorem 2, which implies the existence of a right maximal interval, $[0, \theta_+)$ for $y = y_0(t)$ is either $[0, \infty)$ or $y_0(t) \to \partial E$ as $t \to \theta_+ < \infty$.

The last part of the theorem, that when C^+ has a compact closure in E, then $C_\alpha : y = y_\alpha(t)$ exists on $(-\infty, \infty)$ and $C_\alpha \cup A(C_\alpha) \cup \Omega(C_\alpha) \subset \Omega(C^+)$ follows immediately from (3.4) and the fact that $\Omega(C^+)$ is closed. [6] $\diamond$

Definition 3 [7] A closed, bounded line segment, L in E is called a *transversal* to (3.1) if $f(y) \neq 0$ for $y \in L$ and the direction of $f(y)$ at points $y \in L$ are not parallel to L. All crossings of L by a solution $y = y(t)$ of (3.1) are in the same direction (with respect to increasing t).

3.2 Poincaré-Bendixson

Theorem 5 [8] *(Poincaré-Bendixson)* Let $f(y) = f(y_1, y_2)$ be continuous on an open plane set E, and let $C^+ : y = y_\beta(t)$ be a solution of (3.1) for $t \geq t_0$ with a compact closure in E. In addition, suppose that $y_\beta(t_1) \neq y_\beta(t_2)$ for $0 < t_1 < t_2 < \infty$ and that $\Omega(C^+)$ contains no stationary points. Then

[6] Ibid.
[7] Ibid.
[8] Ibid.

$\Omega(C^+)$ is the set of points $y = (y_1, y_2)$ on a periodic solution $C_p : y = y_p(t)$ of (3.1). Furthermore, if $p > 0$ is the smallest period of $y_p(t)$, then $y_p(t_1) \neq y_p(t_2)$ for $0 \leq t_1 < t_2 < p$; i.e., $J : y = y_p(t),\ 0 \leq t \leq p$ is a Jordan Curve.

Proof. This proof will be divided into sections 1-5.

1 Let $y_\alpha \in E, f(y_\alpha) \neq 0, L$ a transversal through y_α. Then Peano's Existence Theorem implies that there is a small neighborhood E_α of y_α $(E_\alpha \subset E)$ and an $\epsilon > 0$ such that any solution $y = y_\gamma(t)$ of the initial value problem $y' = f(y), y(0) = y_\gamma$ for $y_\gamma \in E_\alpha$ exists for $|t| \leq \epsilon$ and crosses L exactly once for $|t| \leq \epsilon$. In fact, if $\delta > 0$ is arbitrary, E_α and ϵ can be chosen so that $y_\gamma(t)$ exists and differs from $y_\gamma + t \cdot f(y_\alpha)$ by at most $\delta \cdot |t|$ for $|t| \leq \epsilon$. Thus if E_α is sufficiently small, $y = y_\gamma(t)$ crosses L at least once, but can cross L at most once for $|t| \leq \epsilon$ since crossings of L are in the same direction.
In particular, it follows that if $y = y_\gamma(t)$ is a solution of $y' = f$ on a closed, bounded interval, then $y = y_\gamma(t)$ has at most a finite number of crossings of L.

2 Let L be a transversal which (without loss of generality) can be supposed to be on y_2-axis, where $y = (y_1, y_2)$. Suppose that $y = y_\beta(t)$ crosses L at t-values $t_1 < t_2 < \ldots$, then $y_{\beta,2}(t_n)$ is strictly monotone in n.
In order to see this, suppose (again, without loss of generality) that crossings of L occur with increasing y_1 (thus, I can say that y_1 changes from positive to negative as it crosses L). Consider the case that $y_{\beta,2}(t_1) < y_{\beta,2}(t_2)$. The set consisting of the arc $y = y_\beta(t),\ t_1 < t < t_2$, and the line segment $y_{\beta,2}(t_1) < y_2 < y_{\beta,2}(t_2)$ on the y_2-axis forms a Jordan curve, J. For all $t > t_2,\ y = y_\beta(t)$ is in the exterior of J or in the interior of J, by the assumption on $y_\beta(t)$ and the fact that crossings of L occur only in one direction. Thus, this argument can be iterated to show $y_{\beta,2}(t_2) < y_{\beta,2}(t_3)$, etc..

3 It will now be shown that if L is a transversal, $\Omega(C^+)$ contains at most one point on L. For if $y_\alpha \in L \cap \Omega(C^+)$, (1) implies that $y = y_\beta(t)$ crosses L infinitely many times (this occurs whenever $y_\beta(t)$ is near y_α). By (2), the intersections of $y_\beta(t)$ and L tend monotonically to y_α as $t \to \infty$. However, this implies that $L \cap \Omega(C^+)$ cannot contain any point other than y_α.

4 Since C^+ is bounded, $\Omega(C^+)$ is not empty. Let $y_\alpha \in \Omega(C^+)$. By Theorem 4, $y' = f,\ y(0) = y_\alpha$ has a solution, $C_\alpha : y = y_\alpha(t),\ -\infty < t < \infty$,

contained in $\Omega(C^+)$; Thus, $\Omega(C_\alpha) \subset \Omega(C^+)$.
$\Omega(C_\alpha)$ is non-empty. Let $y_\gamma \in \Omega(C_\alpha)$, so that y_γ is a regular point, since $\Omega(C^+)$ contains no stationary points (by assumption). Thus there is a transversal L_γ through y_γ and $y = y_\alpha(t)$ has infinitely many crossings of L_γ near y_γ, but y_γ and every such crossing is a point of $\Omega(C^+)$. By (3), these points coincide. In particular, there exist points $t_1 < t_2$ such that $y_\gamma = y_\alpha(t_1) = y_\alpha(t_2)$. It follows that (3.1) has a periodic solution, $y = y_p(t)$, of period $p = t_2 - t_1$ such that $y_p(t) = y_\alpha(t)$ for $t_1 < t < t_2$. Since $y_\alpha(t)$ is not constant on any t-interval, it can be supposed that $y_p(t_\xi) \neq y_p(t_0)$ for $0 \leq t_0 < t_\xi < p$.

5 It must also be shown that $\Omega(C^+)$ coincides with its subset $C_p : y = y_p(t),\ -\infty < t < \infty$. If not, $\Omega(C^+) - C_p \neq \emptyset$. Then C_p contains a point y_1 which is a cluster point of $\Omega(C^+) - C_p$, since $\Omega(C^+)$ is connected by Theorem 1. Let L_ζ be a transversal through y_ζ. Any small sphere about y_ζ contains points $y_\omega \in \Omega(C^+) - C_p$. For any such $y_\omega,\ y' = f$ has a solution $y = y_\omega(t),\ -\infty < t < \infty$, such that $y_\omega(0) = y_\omega$ and $y_\omega(t)$ is contained in $\Omega(C^+)$ by Theorem 4. If y_ω is sufficiently close to y_ζ, then $y_\omega(t)$ crosses the transversal L_ζ. The crossing is necessarily at the point y_ζ by part 3.
Since $y_\omega \notin C_p$, this is impossible when solutions of initial value problems belonging to (3.1) are unique. I will now show that it is also impossible in the general case.
Let $y_\omega(t_p) \in C_p$, while $y_\omega(t) \notin C_p$ for $t \in (0, t_p)$. Since $y_\omega(t_p)$ is a regular point, there is a transversal L_p through $y_\omega(t_p)$. Then a small translation of L_p in a suitable direction is a transversal, L_{p_α}, which meets C_p and $y = y_\omega(t)$ in two distinct points, since $y_\omega(t) \notin C_p$ for $t \in (0, t_p)$. This contradicts part 3, and thus Poincaré-Bendixson is proved. [9] $\diamond$

[9] Ibid.